Introduction to calculus

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UNIT-1 : Limit and Differentiation

"The concept of a limit is a central idea that distinguishes calculus from algebra and trigonometry. It is fundamental to finding the tangent to a curve or the velocity of an object."

BASIC DEFINITIONS:

- Interval: -A set of the following forms is called as interval
  (1) $\{x \in \mathbb{R} | a \leq x \leq b\} = [a, b]$
  (2) $\{x \in \mathbb{R} | a < x < b\} = (a, b)$
  (3) $\{x \in \mathbb{R} | a \leq x < b\} = [a, b)$
  (4) $\{x \in \mathbb{R} | a < x \leq b\} = (a, b]$

- δ Neighborhood or Neighborhood:- An interval around a point $a \in \mathbb{R}$ is said to be its neighborhood if it is of the form for some $\delta > 0$
  $\{x \in \mathbb{R} | a - \delta < x < a + \delta\} = (a - \delta, a + \delta)$

- Deleted Neighborhood:- An interval around a point $a \in \mathbb{R}$ is said to its be deleted neighborhood if it is of the form for some $\delta > 0$
  $\{x \in \mathbb{R} | a - \delta < x < a + \delta, x \neq a\} = (a - \delta, a + \delta) \sim a$

- Modulus function: A function defined on set of real numbers $\mathbb{R}$ which gives the absolute value(positive value) of a number is called modules function. It is defined as
  $|x| = \begin{cases} x, & \text{if x is non negative;} \\ -x, & \text{if x is negative.} \end{cases}$
  e.g., $|2| = 2, |-2| = 2, |0| = 0$

- Integer Part Function: A function defined on set of real numbers $\mathbb{R}$ which gives the integer part of the number is called integer part function. It is defined as
  $[x] =$ nearest integer less than x.
  e.g., $[3.234] = 3, [-2.234] = -2, [\pi] = 3, [e] = 2.$

- Even function: A function $f(x)$ is said to be even function if $f(-x) = f(x)$, for all x.
  e.g., $f(x) = x^2$

- Odd function: A function $f(x)$ is said to be odd function if $f(-x) = -f(x)$ for all x.
  e.g., $f(x) = x$

1.1 ***Limit: Definition*** Let $f(x)$ be defined on an open interval about x_0, *except possibly at a itself.* If $f(x)$ gets arbitrarily close to L for all x sufficiently close to a, we say that f approaches the **limit** L as x approaches a. Mathematically,

we can write,

$$\lim_{x \to a} f(x) = L$$

The precise definition of Limit:-

The statement

$$\lim_{x\to a} f(x) = L$$

means that for every $\epsilon > 0$ $\exists\delta > o$ such that

$$|f(x) - L| < \epsilon \ whenever \ x \in (a - \delta, a + \delta) \sim \{a\}$$

- ***Right hand limit and Left hand limit:-*** There are two kind of limits over a real line.

  Namely, Right hand limit and Left hand limit.
  Over the line there are two directions to approach any point, from left to the point and from right to the point.
  These limits are defined as follows
  we mean by a left limit

$$\lim_{x\to a_-} f(x) = L$$

For every $\epsilon > 0, \ \exists \ \delta > 0$ such that

$$|f(x) - L| < \epsilon \ \ whenever \ x \in (a - \delta, a)$$

And Similarly,we mean by a right limit

$$\lim_{x\to a_+} f(x) = L$$

For every $\epsilon > 0, \ \exists \ \delta > 0$ such that

$$|f(x) - L| < \epsilon \ \ whenever \ x \in (a, a + \delta)$$

1.2 *Working Rules and Simple Examples of Limit*

(1) $\lim\limits_{x\to a} [kf(x)] = k \ \lim\limits_{x\to a} f(x)$

(2) $\lim\limits_{x\to a} [f(x) + g(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$

(3) $\lim\limits_{x\to a} [f(x) - g(x)] = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$

(4) $\lim\limits_{x\to a} [f(x) \times g(x)] = \lim\limits_{x\to a} f(x) \times \lim\limits_{x\to a} g(x)$

(5) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$, provided that $\lim\limits_{x\to a} g(x) \neq 0$

EXAMPLES:-

(1) $\lim\limits_{x\to 2} (4) = 4$

(2) $\lim\limits_{x\to 2} (5x - 3)$
$= \lim\limits_{x\to 2} (5x) - \lim\limits_{x\to 2} 3 = 5 \times \lim\limits_{x\to 2} x - 3 = 5 \times 2 - 3 = 10 - 3 = 7$

(3) $\lim\limits_{x\to 2} (3x + 2)$
$= \lim\limits_{x\to 2} (3x) + \lim\limits_{x\to 2} 2 = 3 \times \lim\limits_{x\to 2} x + 2 = 3 \times 2 + 2 = 6 + 2 = 8$

1.3 ***Limit of a Polynomial Function:-***

A *Polynomial Function* is of the type

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_{n-1}x^{n-1} + a_nx^n$$

where $a_0, a_1, ..., a_n$ are real numbers.

The limit of the polynomial function is defined as follows

$$\lim_{x\to a} p(x) =$$
$$= \lim_{x\to a}(a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_{n-1}x^{n-1} + a_nx^n)$$
$$= \lim_{x\to a}(a_0) + \lim_{x\to a}(a_1x) + \lim_{x\to a}(a_2x^2) + \lim_{x\to a}(a_3x^3) + ... + \lim_{x\to a}(a_{n-1}x^{n-1}) + \lim_{x\to a}(a_nx^n)$$
$$= \lim_{x\to a}(a_0) + a_1\lim_{x\to a} x + a_2\lim_{x\to a} x^2 + a_3\lim_{x\to a} x^3 + ... + a_{n-1}\lim_{x\to a} x^{n-1} + a_n\lim_{x\to a} x^n$$
$$= a_0 + a_1a + a_2a^2 + a_3a^3 + ... + a_{n-1}a^{n-1} + a_na^n$$
$$= p(a)$$

Hence, the

$$\lim_{x\to a} p(x) = p(a)$$

- # Examples:-

1. Evaluate $\lim_{x\to 2}(3x^2 + x + 1)$

Sol^n: Here we have $p(x) = 3x^2 + x + 1$
And we know that $\lim_{x\to a} p(x) = p(a)$
$\Rightarrow \lim_{x\to 2}(3x^2+x+1) = \lim_{x\to 2}(3x^2)+\lim_{x\to 2}(x)+\lim_{x\to 2}(1) = 3\lim_{x\to 2}(x^2)+\lim_{x\to 2}(x)+1 = 3(2)^2+2+1 = 15$

2. Evaluate $\lim_{x\to 2}(-x^2 + 5x - 2)$

Sol^n: Here, we have $p(x) = -x^2 + 5x - 2$
Hence, $\lim_{x\to 2}(-x^2 + 5x - 2) = -(2)^2 + 5(2) - 2 = -4 + 10 - 2 = 4$

3. Evaluate $\lim_{x\to -2}(x^3 - 2x^2 + 4x+)$

Sol^n: Here, we have $p(x) = x^3 - 2x^2 + 4x + 8$
Hence, $\lim_{x\to -2}(x^3 - 2x^2 + 4x+) = (-2)^3 - 2(-2)^2 + 4(-2) + 8 = -8 - 8 - 8 + 8 = -16$

4. Evaluate $\lim_{x\to 6} 8(t - 5)(t - 7)$

Sol^n: Here we have $p(t) = 8(t - 5)(t - 7) = 8(t^2 - 12t + 35)$ which is a polynomial in t.
Hence, the given limit is $p(6) = 8(6^2 - 12(6) + 35) = 8(36 - 72 + 35) = -8$

1.4 ***Limit of a Rational Function:-***

A rational function is the function of the form $f(x) = \dfrac{p(x)}{q(x)},\ q(x) \neq 0, \forall x$

The limit of the rational function is given by

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{p(x)}{q(x)} = \frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)}, \text{ provided } \lim_{x\to a} q(x) \neq 0 \qquad [\because \text{ By the working rule of limit.}]$$

Further, if $p(x)$ *and* $q(x)$ are polynomials then, we have

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$$

• Examples:-

1. Evaluate $\lim_{x\to 2} \frac{x+3}{x+6}$

Sol^n: Here we have $f(x) = \frac{x+3}{x+6}$ comparing that to the standard form $f(x) = \frac{p(x)}{q(x)}$ of the rational function, we get
$p(x) = x+3$ and $q(x) = x+6$
Using the limit rule of rational function we have,
$$\lim_{x\to 2} \frac{x+3}{x+6} = \frac{\lim_{x\to 2}(x+3)}{\lim_{x\to 2}(x+6)} = \frac{2+3}{2+6} = \frac{5}{8}$$

2. Evaluate $\lim_{x\to 5} \frac{4}{x-7}$

Sol^n: Here we have
$$\lim_{x\to 5} \frac{4}{x-7} = \frac{4}{5-7} = \frac{4}{-2} = -2$$

3. Evaluate $\lim_{y\to -5} \frac{y^2}{5-y}$

Sol^n: Here we have
$$\lim_{y\to -5} \frac{y^2}{5-y} = \frac{(-5)^2}{5-(-5)} = \frac{25}{10} = \frac{5}{2}$$

• **Special type of rational function:-**

$$\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$$

Example 1. Find $\lim_{x\to 1} \frac{x^6-1}{x^8-1}$

Sol^n:- Using the above special method directly.
we get,
The given limit as

$$\lim_{x\to 1} \frac{x^6-1}{x^8-1} = \lim_{x\to 1} \frac{\frac{x^6-1}{x-1}}{\frac{x^8-1}{x-1}} = \frac{6(1)^5}{8(1)^7} = \frac{6}{8}$$

Example 2. Find $\lim\limits_{x\to 2} \dfrac{x^5-32}{\sqrt{x}-\sqrt{2}}$

Sol^n:- Here, the given function is a rational function whose limit is evaluated as

$$\lim_{x\to 2} \frac{x^5-32}{\sqrt{x}-\sqrt{2}} = \frac{\lim\limits_{x\to 2}(x^5-32)}{\lim\limits_{x\to 2}(\sqrt{x}-\sqrt{2})}$$

But, here $\lim\limits_{x\to 2}(x^5-32) = \lim\limits_{x\to 2}(\sqrt{x}-\sqrt{2}) = 0$

Here,

we can rewrite the given limit as follows

$$\lim_{x\to 2} \frac{\frac{(x^5-32)}{(x-2)}}{\frac{(\sqrt{x}-\sqrt{2})}{(x-2)}}$$

$$= \frac{\lim\limits_{x\to 2} \frac{(x^5-2^5)}{(x-2)}}{\lim\limits_{x\to 2} \frac{(x^{1/2}-2^{1/2})}{(x-2)}} = \frac{5(2)^4}{(1/2)(2)^{1/2-1}} = \frac{5\cdot 2^5}{2^{-1/2}} = 5(32)(\sqrt{2}) = 160\sqrt{2}$$

Example 3. Find $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

Sol^n:- Here, the given limit $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

[Multiplying denominator and numerator by conjugate surds]

$$= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{(x+h-x)}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{\lim\limits_{h\to 0}(1)}{\lim\limits_{h\to 0}(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{1}{\sqrt{x+0}+\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Example 4. $\lim\limits_{x\to 2} \dfrac{x^3-3x^2+5x-6}{x^3-8}$

Sol^n:- $\lim\limits_{x\to 2} \dfrac{x^3-3x^2+5x-6}{x^3-8}$

Here, $2^3-3\cdot 2^2+5\cdot 2-6=8-12+10-6=18-18=0$ so $x-2$ is one of its factor.

Hence, x^3-3x^2+5x-6

$=x^3-2x^2-x^2+2x+3x-6$

$=x^2(x-2)-x(x-2)+3(x-2)$

$=(x-2)(x^2-x+3)$

Also, $x^3-8 \Rightarrow\ 2^2-8=0$ hence, $x-2$ is one of its factor.

$x^3-8=(x-2)(x^2+2x+4)$

So, $\lim\limits_{x\to 2} \dfrac{x^3-3x^2+5x-6}{x^3-8}$

$=\lim\limits_{x\to 2} \dfrac{(x-2)(x^2-x+3)}{(x-2)(x^2+2x+4)}$

$=\lim\limits_{x\to 2} \dfrac{x^2-x+3}{x^2+2x+4}$

$=\dfrac{2^2-2+3}{2^2+2\cdot 2+4}=\dfrac{5}{12}$

Example-5. $\lim\limits_{x\to 1} \dfrac{x^6-1}{x^{15}-1} \quad (x\in R-\{1\})$

Sol^n:- Here, for $x^6-1 \Rightarrow\ 1^6-1=0$ and for $x^{15}-1 \Rightarrow\ 1^{15}-1=0$

$x^6-1=(x-1)(x^5+x^4+x^3+x^2+x+1)$

and $x^{15}-1=(x-1)(x^{14}+x^{13}+......+x+1)$

Hence, $\lim\limits_{x\to 1} \dfrac{x^6-1}{x^{15}-1}$

$=\lim\limits_{x\to 1} \dfrac{(x-1)(x^5+x^4+x^3+x^2+x+1)}{(x-1)(x^{14}+x^{13}+......+x+1)}$

$=\dfrac{6(1)^5}{15(1)^{14}}=6/15=3/5$

Example-6. $\lim\limits_{x\to 2} \dfrac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$

Sol^n:- $\lim\limits_{x\to 2} \dfrac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$

$$=\lim_{x\to 2} \frac{(x-2)(x+2)}{\sqrt{x+2}-\sqrt{3x-2}}\cdot\frac{\sqrt{x+2}+\sqrt{3x-2}}{\sqrt{x+2}+\sqrt{3x-2}}$$

$$=\lim_{x\to 2} \frac{(x-2)(x+2)(\sqrt{x+2}+\sqrt{3x-2})}{x+2-3x+2}$$

$$=\lim_{x\to 2} \frac{(x-2)(x+2)(\sqrt{x+2}+\sqrt{3x-2})}{4-2x}$$

$$= \lim_{x\to 2} \frac{(x-2)(x+2)(\sqrt{x+2}+\sqrt{3x-2})}{-2(x-2)}$$

$$= \lim_{x\to 2} \frac{(x+2)(\sqrt{x+2}+\sqrt{3x-2})}{-2}$$

$$= \frac{(2+2)(\sqrt{2+2}+\sqrt{6-2})}{-2} = -4$$

Example-7. $\lim_{x\to 3} \dfrac{x^{\frac{3}{2}} - 3^{\frac{3}{2}}}{x-3}$

Sol.: $\lim_{x\to 3} \dfrac{x^{3/2} - 3^{3/2}}{x-3}$

$$= \frac{3}{2}(3)^{\frac{3}{2}-1} = \frac{3}{2}(3)^{\frac{1}{2}} = \frac{3\sqrt{3}}{2}$$

1.5 *Limit of Trigonometric Function:-*

There are basically two main trigonometric functions.
Namely, sin and cos,

where $\sin\theta$ is defined as ratio of opposite side to the angle θ and hypotonus
and $\cos\theta$ is defined as ratio of adjacent side to the angle θ and hypotonus
Here we consider the following basic results of trigonometry without any justification.

(1) $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$

(2) $\lim_{x\to 0} \dfrac{\tan x}{x} = 1$

(3) $1 + \cos x = 2\cos^2 \frac{x}{2}$

(4) $1 - \cos x = 2\sin^2 \frac{x}{2}$

(5) $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

(6) $\sin^2 x + \cos^2 x = 1$

EXAMPLES :

1. $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$

Sol^n:- Here $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$

$$= \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{2\left(\sin\frac{x}{2}\right)^2}{x^2}$$

$$= \lim_{x\to 0} \quad 2\left(\frac{\sin\frac{x}{2}}{x}\right)^2$$
$$= \lim_{x\to 0} \quad 2\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}\times 2}\right)^2$$
$$= \lim_{x\to 0} \quad \frac{2}{4}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$
$$= \frac{2}{4}\lim_{x\to 0} \quad \left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$
$$= \frac{1}{2} \quad \left(\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$
$$= \frac{1}{2}\ (1)^2$$
$$= \frac{1}{2}$$

2. $\lim\limits_{x\to\frac{\pi}{2}} \quad (\sec x - \tan x)$

Sol^n:- Here $\lim\limits_{x\to\frac{\pi}{2}} \quad (\sec x - \tan x)$

$\lim\limits_{x\to\frac{\pi}{2}} \quad (\sec x - \tan x) = \frac{1}{0} - \frac{1}{0}$, which is not defined so we have to convert given limit in to simple forms, i.e., in terms of sin and cos functions.

$$= \lim_{x\to\frac{\pi}{2}} \quad \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$
$$= \lim_{x\to\frac{\pi}{2}} \quad \left(\frac{1-\sin x}{\cos x}\right)$$
$$= \lim_{x\to\frac{\pi}{2}} \quad \left(\frac{1-\sin x}{\cos x}\right)$$

From the trigonometric identity, $\sin^\theta + \cos^2\theta = 1$, we have

$$\cos x = \sqrt{1-\sin^2 x}$$

Using this the above limit is

$$= \lim_{x\to\frac{\pi}{2}} \quad \left(\frac{1-\sin x}{\sqrt{1-\sin^2 x}}\right)$$
$$= \lim_{x\to\frac{\pi}{2}} \quad \left(\frac{1-\sin x}{\sqrt{(1-\sin x)(1+\sin x)}}\right)$$
$$= \lim_{x\to\frac{\pi}{2}} \quad \frac{\sqrt{1-\sin x}}{\sqrt{1+\sin x}}$$
$$= \lim_{x\to\frac{\pi}{2}} \quad \sqrt{\frac{1-\sin x}{1+\sin x}}$$
$$= \sqrt{\frac{1-1}{1+1}}$$
$$= \sqrt{\frac{0}{2}} = 0$$

3. $\lim_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$

Sol^n:- Here $\lim_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$

$$= \lim_{x\to 0} \frac{\left(\frac{\sin x}{\cos x} - \sin x\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\sin x\left(\frac{1}{\cos x} - 1\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\sin x\left(\frac{1-\cos x}{\cos x}\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\sin x\,(1-\cos x)}{x^3 \cos x}$$

$$= \lim_{x\to 0} \frac{(1-\cos x)}{x^2} \times \lim_{x\to 0} \frac{\sin x}{x} \times \lim_{x\to 0} \frac{1}{\cos x}$$

$$= \lim_{x\to 0} \frac{(1-\cos x)}{x^2} \times 1 \times \frac{1}{1}$$

$$= \lim_{x\to 0} \frac{(1-\cos x)}{x^2}$$

$$= \lim_{x\to 0} \frac{\left(2\sin^2 \frac{x}{2}\right)}{x^2}$$

$$= \lim_{x\to 0} \; 2\left(\frac{\sin\frac{x}{2}}{x}\right)^2$$

$$= \lim_{x\to 0} \; 2\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}\cdot 2}\right)^2$$

$$= \lim_{x\to 0} \frac{2}{4}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$

$$= \frac{2}{4}\lim_{x\to 0} \left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$

$$= \frac{1}{2}(1)^2$$

$$= \frac{1}{2}$$

4. $\lim_{x\to 0} \dfrac{cosecx - \cot x}{x}$

Sol^n:- Here, $\lim_{x\to 0} \dfrac{cosecx - \cot x}{x}$

$$= \lim_{x\to 0} \frac{\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)}{x}$$

$$= \lim_{x\to 0} \frac{\left(\frac{1-\cos x}{\sin x}\right)}{x}$$

$$= \lim_{x\to 0} \frac{\left(\dfrac{2\sin^2\frac{x}{2}}{\sin x}\right)}{x}$$

$$= \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x\dfrac{\sin x}{x} \times x}$$

$$= \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x^2} \times \lim_{x\to 0} \frac{1}{\dfrac{\sin x}{x}}$$

$$= \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x^2} \times \frac{1}{1}$$

$$= \lim_{x\to 0} \frac{\left(2\sin^2\frac{x}{2}\right)}{x^2}$$

$$= \lim_{x\to 0} 2\left(\frac{\sin\frac{x}{2}}{x}\right)^2$$

$$= \lim_{x\to 0} 2\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}\cdot 2}\right)^2$$

$$= \lim_{x\to 0} \frac{2}{4}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$

$$= \frac{2}{4}\lim_{x\to 0} \left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$

$$= \frac{1}{2}(1)^2$$

$$= \frac{1}{2}$$

5. $\lim_{x\to\pi} \dfrac{1+\cos x}{\tan^2 x}$

Sol^n:- Here, $\lim_{x\to\pi} \dfrac{1+\cos x}{\tan^2 x}$

$$= \lim_{x\to\pi} \frac{1+\cos x}{\frac{\sin^2 x}{\cos^2 x}}$$

$$= \lim_{x\to\pi} \frac{(1+\cos x)\cos^2 x}{\sin^2 x}$$

$$= \lim_{x\to\pi} \frac{(1+\cos x)\cos^2 x}{1-\cos^2 x}$$

$$= \lim_{x\to\pi} \frac{(1+\cos x)\cos^2 x}{(1-\cos x)(1+\cos x)}$$

$$= \lim_{x\to\pi} \frac{\cos^2 x}{(1-\cos x)}$$

$$= \frac{(-1)^2}{(1-(-1))}$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

1.6 ***Differentiation: Definition***

Consider $y = f(x)$, a function of independent variable x, then its differentiation with respect x is defined as a limit

$\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$. And it is denoted by any of the notations $f'(x), \dfrac{dy}{dx}, \dfrac{df}{dx}$ *or* y_1

1.7 ***Simple examples of Differentiation***

Ex.1 Obtain $\dfrac{d}{dx}x^4$ by definition.

Sol^n:- Here $f(x) = x^4$,

$$\frac{d}{dx}x^4 = \lim_{h\to 0}\frac{(x+h)^4 - x^4}{h}$$
$$= \lim_{h\to 0}\frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$
$$= \lim_{h\to 0}\frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$
$$= \lim_{h\to 0} 4x^3 + 6x^2h + 4xh^2 + h^3$$
$$= 4x^3$$

Ex.2 Obtain $\dfrac{d}{dx}(\sqrt{x})$ by definition.

Sol^n:- Here $f(x) = \sqrt{x}$,

$$\frac{d}{dx}(\sqrt{x}) = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

[Multiplying denominator and numerator by conjugate surds]

$$= \lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$
$$= \lim_{h\to 0}\frac{(x+h-x)}{h(\sqrt{x+h}+\sqrt{x})}$$
$$= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$
$$= \frac{\lim\limits_{h\to 0}(1)}{\lim\limits_{h\to 0}(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{1}{\sqrt{x+0}+\sqrt{x}}$$
$$= \frac{1}{2\sqrt{x}}$$

- ***Formulas of Derivatives of Trigonometric Functions:-***

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$cosecx$	$-cosecx \cot x$
$\cot x$	$-cosec^2 x$

- ***Formulas of Derivatives of other standard Functions:-***

$f(x)$	$f'(x)$
$\log x$	$\frac{1}{x}$
e^x	e^x
a^x	$a^x \log a$

1.8 *Working Rules of Differentiation*

- **Derivative of constant**

If f has the constant value $f(x) = c$, then
$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Examples:-

1. If $f(x) = 8$, then
$$\frac{df}{dx} = \frac{d}{dx}(8) = 0.$$

2. If $f(x) = -\frac{\pi}{2}$ then
$\frac{df}{dx} = \frac{d}{dx}\left(-\frac{\pi}{2}\right) = 0.$

- **Power Rule of derivative.**

If n the any real number for $f(x) = x^n$, then
$$\frac{df}{dx} = \frac{d}{dx}(x^n) = nx^{n-1}.$$

Examples:-

1. Interpreting the above Rule

$f(x)$	x^{-2}	x^{-1}	x^0	x^1	x^2	....
$f'(x)$	$-2x^{-3}$	$-x^{-2}$	0	1	$2x$	....

2. Find $\frac{d}{dx}x^{5/2}$ and $\frac{d}{dx}x^{7/2}$.

Here, by using the power rule of derivative, we have,

$$\frac{d}{dx}x^{5/2} = \tfrac{5}{2}x^{5/2-1} = \tfrac{5}{2}x^{3/2}$$

Similarly,

$$\frac{d}{dx}x^{7/2} = \tfrac{7}{2}x^{7/2-1} = \tfrac{7}{2}x^{5/2}$$

- **Constant Multiple Rule.**

If $u(x)$ is a differentiable function, and c is a constant,then
$$\frac{d(cu)}{dx} = \frac{d}{dx}(cu)) = c\frac{du}{dx}.$$

Examples:-

1. Differentiate $f(x) = 2x^5$.
Here, $f(x)$ is the constant multiple(i.e., 2) of the function x^5,
$\Rightarrow \frac{d}{dx}f(x) = \frac{d}{dx}(2x^5) = 2\frac{d}{dx}x^5 = 2\cdot(5x^4)$ [$\because$ By using the power rule]

- **Derivative Sum Rule**

> *If u and v are differentiable functions of x, then their sum is differentiable and its derivative is given by the rule*
> $$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Examples:-

1. Find the derivative of $y = x^4 + 12x$

   Here, $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^4 + 12x)$

   $= \dfrac{d}{dx}(x^4) + \dfrac{d}{dx}(12x)$ [Sum Rule of Derivative]

   $= 4x^3 + 12\dfrac{d}{dx}(x)$ [By Power Rule]

   $= 4x^3 + 12$ [By Power Rule]

2. ***Derivative of Polynomial Function:-***

   Consider the standard form of the polynomial function
   $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n$
   whose derivative is given by
   $p'(x) = \dfrac{d}{dx}(p(x)) = \dfrac{d}{dx}(a_0) + \dfrac{d}{dx}(a_1x) + \dfrac{d}{dx}(a_2x^2) + \dfrac{d}{dx}(a_3x^3) + ... + \dfrac{d}{dx}(a_nx^n)$

   $p'(x) = \dfrac{d}{dx}(p(x)) = \dfrac{d}{dx}(a_0) + a_1\dfrac{d}{dx}(x) + a_2\dfrac{d}{dx}(x^2) + a_3\dfrac{d}{dx}(x^3) + ... + a_n\dfrac{d}{dx}(x^n)$

   . [$\because$ By using the constant multiple rule]

   $p'(x) = \dfrac{d}{dx}(p(x)) = (0) + a_1(1) + a_2(2x) + a_3(3x^2) + ... + a_n(nx^{n-1})$

   . [$\because$ By using the power rule]

   $p'(x) = \dfrac{d}{dx}(p(x)) = a_1 + 2a_2x + 3a_3x^2 + ... + na_nx^{n-1}$

3. Find $\dfrac{d}{dx}\left(x^3 + \frac{4}{3}x^2 - 5x + 1\right)$

   $\dfrac{d}{dx}\left(x^3 + \frac{4}{3}x^2 - 5x + 1\right) = \dfrac{d}{dx}(x^3) + \dfrac{d}{dx}\left(\frac{4}{3}x^2\right) - 5\dfrac{d}{dx}(x) + \dfrac{d}{dx}(1) = 3x^2 + \frac{8}{3}x - 5$

- **Derivative of Product**

> *If u and v are two differentiable functions, then*
> $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Examples:-

1. Differentiate $y = (x^2+1)(x^3+3)$
   Here, $y = (x^2+1)(x^3+3)$
   $\Rightarrow \dfrac{dy}{dx} = \dfrac{d}{dx}\{(x^2+1)(x^3+3)\}$
   $= \{(x^2+1)\dfrac{d}{dx}(x^3+3) + \dfrac{d}{dx}(x^2+1)(x^3+3)\}$
   $= (x^2+1)(3x^2) + (2x)(x^3+3)$
   $= 3x^4 + 3x^2 + 2x^4 + 6x$
   $= 5x^4 + 3x^2 + 6x$

2. Differentiate $y = (x-1)(x^2+x+1)$
   Here, $y = (x-1)(x^2++x+1)$
   $\Rightarrow \dfrac{dy}{dx} = \dfrac{d}{dx}\{(x-1)(x^2+x+1)\}$
   $= \{(x-1)\dfrac{d}{dx}(x^2+x+1) + \dfrac{d}{dx}(x-1)(x^2+x+1)\}$
   $= (x-1)(2x) + (1)(x^2+x+1)$
   $= 2x^2 - 2x + x^2 + x + 1$
   $= 3x^2 - x + 1$

- **Derivative of Quotient**

> *If u and v are two differentiable functions, then*
> $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Examples:-

- Find $\dfrac{d}{dx}\left(\dfrac{x}{1-x}\right))$

  Here,
  $\dfrac{d}{dx}\left(\dfrac{x}{1-x}\right)$
  $= \dfrac{\frac{d}{dx}(x)(1-x) - x\frac{d}{dx}(1-x)}{(1-x)^2}$
  $= \dfrac{1\cdot(1-x) - x(-1)}{(1-x)^2} = \dfrac{1}{(1-x)^2}$

1.9 *Chain Rules of Differentiation (Derivative of a Composite Function:)*

"In this section we are going to discuss about the differentiation of the functions which are combination(mathematically composition) of two or more than two functions. e.g., $\sqrt{\frac{1-x}{1+x}}$, $\sin(x^2+1)$"

Definition: ***Composite function:-***
If $f : A \to B$ and $g : B \to C$ then their composition $gof : A \to C$ is defined as $gof(x) = g(f(x))$

Note:- Here we can not define the composition fog

Now, we see that how to find the derivative of such function.

If $f(x)$ and $g(x)$ are to differentiable functions and their composition gof is defined then its derivative is given by the following formula

$$\boxed{\frac{d}{dx}[gof] = g'(f(x)) \cdot f'(x)}$$

For problems points of view, one can consider new label for inner function in this case $f(x)$,
let $f(x) = u$ this gives
$gof(x) = g(f(x)) = g(u)$
Here, $\frac{d}{dx}[gof(x)] = \frac{d}{dx}[g(u)] = \frac{d}{du}\{g(u)\} \cdot \frac{du}{dx}$

Examples:-

1. Find $\frac{dy}{dx}$ if $y = \sin x^2$.
Here,
Let us denote $u = x^2$

So, $y = \sin u$ and $u = x^2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d(\sin u)}{du}\frac{d(x^2)}{dx}$$

$$= \cos u \cdot 2x = 2x \cdot \cos x^2$$

2. Find $\frac{dy}{dx}$ if $y = \sin^2 x$.
Here,
Let us denote $y = (\sin x)^2 = u^2$

So, $y = u^2$ and $u = \sin x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d(u^2)}{du}\frac{d(\sin x)}{dx}$$
$$= 2u \cdot \cos x = 2\sin x \cos x$$

- Find derivative of following function w.r.t x.

Ex:-1. $x^3 + 3^x + 3^3$

Sol^n:- Here, we want to find $\frac{d}{dx}(x^3 + 3^x + 3^3)$

$$\frac{d}{dx}(x^3 + 3^x + 3^3)$$
$$= \frac{d}{dx}(x^3) + \frac{d}{dx}(3^x) + \frac{d}{dx}(3^3)$$
$$= \frac{d}{dx}(x^3) + \frac{d}{dx}(3^x) + \frac{d}{dx}(3^3)$$
$$= 3x^2 + 3^x \log 3 + 0$$
$$= 3x^2 + 3^x \log 3$$

Ex:-2. $x \sin x$

Sol^n:- Here, we want to find $\frac{d}{dx}(x \sin x)$

$$\frac{d}{dx}(x \sin x)$$
$$= \frac{d}{dx}(x)\sin x + x\frac{d}{dx}(\sin x) \qquad [\because \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}]$$
$$= 1 \cdot \sin x + x\cos x$$
$$= \sin x + x\cos x$$

Ex:-3 $\frac{\tan x}{x}$

Sol^n:- Here, we want to find $\frac{d}{dx}\left(\frac{\tan x}{x}\right)$

$$\frac{d}{dx}\left(\frac{\tan x}{x}\right)$$
$$= \frac{x\frac{d}{dx}(\tan x) - \tan x\frac{d}{dx}(x)}{x^2} \qquad [\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.]$$
$$= \frac{x\sec^2 x - \tan x}{x^2}$$

Ex-4 $e^{2x}(e^x - e^{-x})$

Sol^n:- Here, we have to find $\frac{d}{dx}(e^{2x}(e^x - e^{-x}))$

$$= \frac{d}{dx}(e^{2x}e^x - e^{2x}e^{-x})$$
$$= \frac{d}{dx}(e^{3x}) - \frac{d}{dx}(e^x)$$

$$= e^{3x}\frac{d}{dx}(3x) - e^x$$

$$= 3e^{3x} - e^x$$

Ex:-5 $\log_{10}(x^2+1)$

Sol^n:- Here, we have to find $\frac{d}{dx}(\log_{10}(x^2+1))$

$$\frac{d}{dx}(\log_{10}(x^2+1))$$

$$= \frac{d}{dx}\left(\frac{\log_e(x^2+1)}{\log_e 10}\right)$$

$$= \frac{1}{\log_e 10}\ \frac{d}{dx}\left(\log_e(x^2+1)\right)$$

$$= \frac{1}{\log_e 10}\ \left(\frac{1}{x^2+1}\right)\frac{d}{dx}(x^2+1)$$

$$= \frac{1}{\log_e 10}\ \left(\frac{2x}{x^2+1}\right)$$

Ex:-6 e^{ax}

Sol^n:- Here, we have to find $\frac{d}{dx}(e^{ax})$

$$\frac{d}{dx}(e^{ax}) = e^{ax}\frac{d}{dx}(ax) = e^{ax}\ a = ae^{ax}$$

EX:-7 $\sin^3 x$

Sol^n:- Here, we want to find $\frac{d}{dx}(\sin^3 x)$

$$\frac{d}{dx}(\sin^3 x)$$

$$= 3\sin^2 x \cdot \frac{d}{dx}(\sin x)$$

$$= 3\sin^2 x \cdot \cos x$$

Ex:-8 $\frac{\tan 3x}{3^x}$

Sol^n:- Here, we want to find $\frac{d}{dx}\left(\frac{\tan 3x}{3^x}\right)$

$$\frac{d}{dx}\left(\frac{\tan 3x}{3^x}\right)$$

$$= \frac{3^x\frac{d}{dx}(\tan 3x) - \tan 3x\frac{d}{dx}(3^x)}{(3^x)^2} \qquad [\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.]$$

$$= \frac{3^x(\sec^2 3x \frac{d}{dx}(3x)) - \tan 3x(3^x \log 3)}{(3^x)^2}$$
$$= \frac{3^x \cdot 3(\sec^2 3x) - \tan 3x(3^x \log 3)}{(3^x)^2}$$
$$= \frac{3^x(3\sec^2 3x - \tan 3x \log 3)}{(3^x)^2}$$
$$= \frac{(3\sec^2 3x - \tan 3x \log 3)}{3^x}$$

1.10 ***Derivative of the Inverse Functions:-***

Here we consider the following two formulas:-

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

Ex:-1 Find $\frac{d}{dx}\left(\sin^{-1} \frac{x}{a}\right)$

Sol^n : $-$ Here, we want to find

$$\frac{d}{dx}\left(\sin^{-1} \frac{x}{a}\right)$$
$$= \frac{1}{\sqrt{1-(\frac{x}{a})^2}} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)$$
$$= \frac{1}{\sqrt{1-(\frac{x^2}{a^2})}} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)$$
$$= \frac{a}{\sqrt{a^2-x^2}} \cdot \frac{1}{a}\frac{d}{dx}(x)$$
$$= \frac{1}{\sqrt{a^2-x^2}}$$

Ex:-2 Find $\frac{d}{dx}(\cos^{-1}(4x^3-3x))$

Sol^n : $-$ Here, we want to find

$$\frac{d}{dx}(\cos^{-1}(4x^3-3x))$$
$$= \frac{-1}{\sqrt{1-(4x^3-3x)^2}} \quad \frac{d}{dx}(4x^3-3x)$$
$$= \frac{-1}{\sqrt{1-(4x^3-3x)^2}} \quad (\frac{d}{dx}(4x^3) - \frac{d}{dx}(3x))$$
$$= \frac{-1}{\sqrt{1-(4x^3-3x)^2}} \quad (4\frac{d}{dx}(x^3) - 3\frac{d}{dx}(x))$$

$$= \frac{-1}{\sqrt{1-(4x^3-3x)^2}} \quad (4(3x^2)-3)$$
$$= \frac{-3(4x^2-1)}{\sqrt{1-(4x^3-3x)^2}}$$

1.11 *Derivative of an Implicit Functions:-*

Sometimes when y is a function of x which can not be explicitly given in the form $y = f(x)$, but they are related by $F(x, y) = 0$. i.e., $F(x, f(x)) = 0$.
e.g., $x^2 + y^2 = xy$.

Exercise:- Find $\frac{dy}{dx}$, if

Ex.1. $x^3 + y^3 = 3axy$

Sol^n : $-$ Here we want find $\frac{dy}{dx}$ from the given equation $x^3 + y^3 = 3axy$

Applying $\frac{d}{dx}$ on both sides of the equation $x^3 + y^3 = 3axy$, we get

$$\Rightarrow \frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(3axy)$$
$$\Rightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 3a\frac{d}{dx}(xy)$$
$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 3a\left(\frac{d}{dx}(x)y + x\frac{d}{dx}(y)\right)$$
$$\Rightarrow x^2 + y^2\frac{dy}{dx} = a\left(1 \cdot y + x\frac{dy}{dx})\right)$$
$$\Rightarrow x^2 + y^2\frac{dy}{dx} = a\left(y + x\frac{dy}{dx})\right)$$
$$\Rightarrow y^2\frac{dy}{dx} - ax\frac{dy}{dx} = ay - x^2$$
$$\Rightarrow y^2\frac{dy}{dx} - ax\frac{dy}{dx} = ay - x^2$$
$$\Rightarrow (y^2 - ax)\frac{dy}{dx} = (ay - x^2)$$
$$\Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

Ex.2. $x + y = \sin xy$

Sol^n : $-$ Here we want find $\frac{dy}{dx}$ from the given equation $x + y = \sin xy$

Applying $\frac{d}{dx}$ on both sides of the equation $x + y = \sin xy$, we get

$$\Rightarrow \frac{d}{dx}(\text{x+y}) = \frac{d}{dx}(\sin xy)$$
$$\Rightarrow \frac{d}{dx}(x) + \frac{d}{dx}(y) = \cos xy\frac{d}{dx}(xy)$$

$$\Rightarrow \frac{d}{dx}(x) + \frac{d}{dx}(y) = \cos xy \frac{d}{dx}(xy)$$
$$\Rightarrow 1 + \frac{dy}{dx} = \cos xy \left(\frac{d}{dx}(x)y + x\frac{d}{dx}(y) \right)$$
$$\Rightarrow 1 + \frac{dy}{dx} = y\cos xy \frac{d}{dx}(x) + x\cos xy \frac{d}{dx}(y)$$
$$\Rightarrow 1 + \frac{dy}{dx} = y\cos xy + x\cos xy \frac{dy}{dx}$$
$$\Rightarrow 1 + \frac{dy}{dx} = y\cos xy + x\cos xy \frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx} - x\cos xy \frac{dy}{dx} = y\cos xy - 1$$
$$\Rightarrow \frac{dy}{dx}(1 - x\cos xy) = y\cos xy - 1$$
$$\Rightarrow \frac{dy}{dx} = \frac{y\cos xy - 1}{1 - x\cos xy}$$

Ex.3. $e^x + e^y = e^{x+y}$

Sol^n : $-$ Here we want find $\frac{dy}{dx}$ from the given equation $e^x + e^y = e^{x+y}$

Applying $\frac{d}{dx}$ on both sides of the equation $e^x + e^y = e^{x+y}$, we get

$$\Rightarrow \frac{d}{dx}(e^x + e^y) = \frac{d}{dx}(e^{x+y})$$
$$\Rightarrow \frac{d}{dx}(e^x) + \frac{d}{dx}(e^y) = e^{x+y}\frac{d}{dx}(x+y)$$
$$\Rightarrow e^x + e^y\frac{dy}{dx} = e^{x+y}\left(\frac{d}{dx}(x) + \frac{d}{dx}(y)) \right)$$
$$\Rightarrow e^x + e^y\frac{dy}{dx} = e^{x+y}\left(1 + \frac{dy}{dx} \right)$$
$$\Rightarrow e^y\frac{dy}{dx} - e^{x+y}\frac{dy}{dx} = e^{x+y} - e^x$$
$$\Rightarrow (e^y - e^{x+y})\frac{dy}{dx} = e^{x+y} - e^x$$
$$\Rightarrow \frac{dy}{dx} = \frac{e^{x+y} - e^x}{e^y - e^{x+y}}$$

1.12 ***Derivative of a Function in Parametric form:-***

A form of the function $F(x, y) = 0$ by the equations $x = f(t)$ *and* $y = g(t)$ is known as parametric form.
Suppose $x = f(t)$ and $y = g(t)$ both are differentiable.
Then

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

e.g., (i) $x = t,\ y = t^2 + 1$, (ii) $x = \sin t,\ y = \cos t$

Exercise:- Find $\frac{dy}{dx}$ for the following:

Ex.1. $x = \cos^3 t, y = sin^3 t. \qquad t \in (0, \frac{\pi}{2})$

Sol^n : $-$ Here, the given equations are in parametric form, t is a parameter.

$$x = \cos^3 t, \quad y = \sin^3 t$$

So, here $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Next,$y = \sin^3 t \ \Rightarrow \frac{dy}{dt} = \frac{d}{dt}(\sin^3 t) = 3\sin^2 t \cdot \frac{d}{dt}(\sin t) = 3\sin^2 t \cdot \cos t$

And

$x = \cos^3 t \ \Rightarrow \frac{dx}{dt} = \frac{d}{dt}(\cos^3 t) = 3\cos^2 t \cdot \frac{d}{dt}(\cos t) = 3\cos^2 t \cdot (-\sin t) = -3\cos^2 t \cdot \sin t$

So, $\frac{dy}{dx} = \frac{3\sin^2 t \cdot \cos t}{-3\cos^2 t \cdot \sin t} = \frac{\sin t}{-\cos t} = -\tan t$

Ex.2. $x = a(1 - \cos\theta), y = a(\theta - \sin\theta). \qquad a \neq 0, \theta \in (0, \pi)$

Sol^n : $-$ Here, the given equations are in parametric form, θ is a parameter.

$$x = a(1 - \cos\theta), y = a(\theta - \sin\theta)$$

So, here $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

Next,$y = a(\theta - \sin\theta) \ \Rightarrow \frac{dy}{dt} = \frac{d}{dt}(a(\theta - \sin\theta)) = a(1 - \cos\theta)$

And

$x = a(1 - \cos\theta) \ \Rightarrow \frac{dx}{dt} = \frac{d}{dt}(a(1 - \cos\theta)) = a(0 - (-\sin\theta)) = a\sin\theta$

So, $\frac{dy}{dx} = \frac{a(1 - \cos\theta)}{a\sin\theta} = \frac{1 - \cos\theta}{\sin\theta}$

1.13 *Exponential Differentiation:-*

We have noted earlier that in the formula table that

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

Can we apply the formula to get $\frac{d}{dx}\left[e^{\sin x}\right]$?
The answer is yes, we can do so.

Let see that as follows:

Example:- Let $y = \left[e^{\sin x}\right]$ then find $\frac{dy}{dx}$.

In the present example two functions are composed, namely exponential function (e^x) and $\sin x$

So,
$$\frac{dy}{dx} = \frac{d}{dx}\left[e^{\sin x}\right] = e^{\sin x} \cdot \frac{d}{dx}(\sin x) = e^{\sin x} \cdot \cos x = \cos x \cdot e^{\sin x}$$

1.14 *Logarithmic Differentiation:-*

We have noted earlier that in the formula table that

$$\boxed{\frac{d}{dx}(\log x) = \frac{1}{x}}$$

Can we apply the formula to get $\frac{d}{dx}\left[(x)^{\sin x}\right]$?
The answer is yes, we can do so.

Let see that as follows:

Exercise: Find the following:

Ex.1. $\frac{d}{dx}\left(x^{\sin x}\right)$

Sol^n : $-$ Here, we have to use "Log" to get the derivative.

First we let $y = x^{\sin x}$

Take 'log' on both sides, we get,

$\Rightarrow \log y = \log(x)^{\sin x}$

$\Rightarrow \log y = \sin x \log x$

Differentiating with respect to x, i.e., applying $\frac{d}{dx}$ on both sides, we get,

$$\Rightarrow \frac{d}{dx}(\log y) = \frac{d}{dx}(\sin x \log x)$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \sin x \frac{1}{x} + \log x(\cos x)$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{x} + \cos x\ \ \log x$$
$$\Rightarrow \frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x\ \ \log x\right]$$
$$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x\ \ \log x\right]$$

Ex.2. $\frac{d}{dx}\left((\sin x)^x + x^{\cos x}\right)$

Sol^n : $-$ Here, we take $y = (\sin x)^x + x^{\cos x}$

and $u = (\sin x)^x$, $\quad v = x^{\cos x}$

Hence, $y = u + v$

We know that $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

So, it is enough to find $\frac{du}{dx}$ *and* $\frac{dv}{dx}$

Here, we have to use "Log" to get these derivatives.

First we let $u = (\sin x)^x$

Take 'log' on both sides, we get,

$\Rightarrow \log u = \log(\sin x)^x$

$\Rightarrow \log u = x \log \sin x$

Differentiating with respect to x, i.e., applying $\frac{d}{dx}$ on both sides, we get,

$$\Rightarrow \frac{d}{dx}(\log u) = \frac{d}{dx}(x \log \sin x)$$
$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}(\log \sin x) + \log \sin x \frac{d}{dx}(x)$$
$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x\frac{1}{\sin x}\frac{d}{dx}(\sin x) + \log \sin x \cdot 1$$
$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{x \cos x}{\sin x} + \log \sin x$$
$$\Rightarrow \frac{du}{dx} = u\left[\frac{x \cos x}{\sin x} + \log \sin x\right]$$
$$\Rightarrow \frac{du}{dx} = (\sin x)^x \left[x \cot x + \log \sin x\right]$$

Next, we let $v = x^{\cos x}$

Take 'log' on both sides, we get,

$\Rightarrow \log v = \log x^{\cos x}$

$\Rightarrow \log v = \cos x \log x$

Differentiating with respect to x, i.e., applying $\dfrac{d}{dx}$ on both sides, we get,

$$\Rightarrow \frac{d}{dx}(\log v) = \frac{d}{dx}(\cos x \log x)$$
$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \cos x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\cos x)$$
$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \cos x \frac{1}{x} + \log x \cdot (-\sin x)$$
$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{\cos x}{x} - \sin x \cdot \log x$$
$$\Rightarrow \frac{dv}{dx} = v\left[\frac{\cos x}{x} - \sin x \cdot \log x\right]$$
$$\Rightarrow \frac{dv}{dx} = x^{\cos x}\left[\frac{\cos x}{x} - \sin x \cdot \log x\right]$$

Hence, $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^x\left[x \cot x + \log \sin x\right] + x^{\cos x}\left[\frac{\cos x}{x} - \sin x \cdot \log x\right]$$

Ex.3. $\dfrac{d}{dx}\left(x^{\sqrt{x}} + (\sqrt{x})^x\right) \quad x > 0.$

Sol^n : $-$ Here, we take $y = x^{\sqrt{x}} + x^{\cos x}$

and $u = x^{\sqrt{x}}, \quad v = (\sqrt{x})^x$

Hence, $y = u + v$

We know that $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$

So, it is enough to find $\dfrac{du}{dx}$ *and* $\dfrac{dv}{dx}$

Here, we have to use "Log" to get these derivatives.

First we let $u = x^{\sqrt{x}}$

Take 'log' on both sides, we get,

$\Rightarrow \log u = \log x^{\sqrt{x}}$

$\Rightarrow \log u = \sqrt{x} \log x$

Differentiating with respect to x, i.e., applying $\dfrac{d}{dx}$ on both sides, we get,

$$\Rightarrow \frac{d}{dx}(\log u) = \frac{d}{dx}(\sqrt{x} \log x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \sqrt{x}\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(\sqrt{x})$$
$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \sqrt{x}\frac{1}{x} + \log x \cdot \frac{1}{2\sqrt{x}}$$
$$\Rightarrow \frac{1}{u}\frac{du}{dx} = 1 + \frac{\log x}{2\sqrt{x}}$$
$$\Rightarrow \frac{du}{dx} = u\left[1 + \frac{\log x}{2\sqrt{x}}\right]$$
$$\Rightarrow \frac{du}{dx} = x^{\sqrt{x}}\left[1 + \frac{\log x}{2\sqrt{x}}\right]$$

Next, we let $v = (\sqrt{x})^x$

Take 'log' on both sides, we get,

$$\Rightarrow \log v = \log(\sqrt{x})^x$$

$$\Rightarrow \log v = x \log\sqrt{x} = x \cdot \frac{1}{2}\log x$$

Differentiating with respect to x, i.e., applying $\frac{d}{dx}$ on both sides, we get,

$$\Rightarrow \frac{d}{dx}(\log v) = \frac{1}{2}\frac{d}{dx}(x\log x)$$
$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{2}\left[x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x)\right]$$
$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{2}\left[x\frac{1}{x} + \log x \cdot 1\right]$$
$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{2}\left[1 + \log x\right]$$
$$\Rightarrow \frac{dv}{dx} = v\frac{1}{2}\left[1 + \log x\right]$$
$$\Rightarrow \frac{dv}{dx} = \frac{(\sqrt{x})^x}{2}\left[1 + \log x\right]$$

Hence, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$$\Rightarrow \frac{dy}{dx} = x^{\sqrt{x}}\left[1 + \frac{\log x}{2\sqrt{x}}\right] + \frac{(\sqrt{x})^x}{2}\left[1 + \log x\right]$$

EXAMPLE: If $x^y = e^{x-y}$ then prove that $\frac{dy}{dx} = \frac{\log x}{[\log xe]^2}$

Solution: Here, we need to take log on both side of the given equation $x^y = e^{x-y}$

i.e., $\log x^y = \log e^{x-y}$

$$\Rightarrow y\log x = (x-y)\log e$$

$\Rightarrow y\log x = (x-y)$ [Because log e=1]

$$\Rightarrow y \log x \ + y = x$$

$$\Rightarrow y(\log x \ + 1) = x$$

$$\Rightarrow y = \frac{x}{(\log x \ + 1)}$$

Applying $\frac{d}{dx}$ on both sides we get,

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(x) \cdot (\log x + \ 1) - x\frac{d}{dx}(\log x + \ 1)}{(\log x + \ 1)^2}$$

$$\frac{dy}{dx} = \frac{1 \cdot (\log x + \ 1) - x\frac{1}{x}}{(\log x + \ 1)^2}$$

$$\frac{dy}{dx} = \frac{\log x + \ 1 - 1}{(\log x + \ 1)^2}$$

$$\frac{dy}{dx} = \frac{\log x}{(\log x + \ 1)^2}$$

$$\frac{dy}{dx} = \frac{\log x}{(\log x + \log e)^2}$$

$$\frac{dy}{dx} = \frac{\log x}{(\log xe)^2}$$

EXTRA EXAMPLES

Ex-1. Find $\lim_{x\to 2} \frac{\sqrt[3]{x+6} - \sqrt[3]{2x+4}}{x^2-4}$

Sol^n:- Here, the given limit

$$\lim_{x\to 2} \frac{\sqrt[3]{x+6} - \sqrt[3]{2x+4}}{x^2-4}$$

$$= \lim_{x\to 2} \frac{\sqrt[3]{x+6} - \sqrt[3]{2x+4}}{x^2-4}$$

$\because$ Multiplying denominator and numerator by term"$(x+6)^{2/3} + (x+6)^{1/3}(2x+4)^{1/3} + (2x+4)^{1/3}$" And using the result $\underbrace{(a-b)(a^2+ab+b^2) = a^3 - b^3}$

$$= \lim_{x\to 2} \frac{\sqrt[3]{x+6} - \sqrt[3]{2x+4}}{x^2-4} \cdot \frac{((x+6)^{2/3} + (x+6)^{1/3}(2x+4)^{1/3} + (2x+4)^{2/3})}{((x+6)^{2/3} + (x+6)^{1/3}(2x+4)^{1/3} + (2x+4)^{2/3})}$$

$$= \lim_{x\to 2} \frac{x+6-(2x+4)}{x^2-4} \cdot \frac{1}{((x+6)^{2/3} + (x+6)^{1/3}(2x+4)^{1/3} + (2x+4)^{2/3})}$$

$$= \lim_{x\to 2} \frac{-x+2}{(x-2)(x+2)} \cdot \frac{1}{((x+6)^{2/3} + (x+6)^{1/3}(2x+4)^{1/3} + (2x+4)^{2/3})}$$

$$= \lim_{x\to 2} \frac{1}{-(x+2)} \cdot \frac{1}{((x+6)^{2/3} + (x+6)^{1/3}(2x+4)^{1/3} + (2x+4)^{2/3})}$$

$$= \lim_{x\to 2} \frac{1}{-(x+2)} \cdot \lim_{x\to 2} \frac{1}{((x+6)^{2/3} + (x+6)^{1/3}(2x+4)^{1/3} + (2x+4)^{2/3})}$$

$$= \frac{1}{-(2+2)} \cdot \frac{1}{((2+6)^{2/3} + (2+6)^{1/3}(2\cdot 2+4)^{1/3} + (2\cdot 2+4)^{2/3})}$$

$$= \frac{1}{-4} \cdot \frac{1}{((8)^{2/3} + (8)^{1/3}(8)^{1/3} + (8)^{2/3})}$$

$$= \frac{1}{-4} \cdot \frac{1}{(4+4+4)} = -\frac{1}{4(12)} = -\frac{1}{48}$$

Ex-2. Find $\lim_{x\to 2} \dfrac{x^3-3x^2-2x+8}{2x^3-3x-10}$

Sol^n:- Here, the given function is a rational function whose limit is evaluated as

Observe that $\lim_{x\to 2} \dfrac{x^3-3x^2-2x+8}{2x^3-3x-10} = \dfrac{\lim_{x\to 2}(x^3-3x^2-2x+8)}{\lim_{x\to 2}(2x^3-3x-10)}$

But, we can see that

$\lim_{x\to 2}(x^3-3x^2-2x+8) = 2^3 - 3(2)^2 - 2(2) + 8 = 8 - 12 - 4 + 8 = 16 - 16 = 0$

$\lim_{x\to 2}(2x^3-3x-10) = 2(2)^3 - 3(2) - 10 = 16 - 6 - 10 = 0$

Next, we can factorize the polynomials $x^3-3x^2-2x+8, 2x^3-3x-10$ as follows

x^3-3x^2-2x+8

$= (x^3 - \underline{2x^2 - x^2} + \underline{2x - 4x} + 8)$

$= x^2\underline{(x-2)} - x\ \underline{(x-2)} - 4\ \underline{(x-2)} = (x-2)(x^2-x-4)$

And similarly,

$2x^3-3x-10$

$= 2x^3 - 4x^2 + 4x^2 - 8x + 5x - 10$

$= 2x^2(x-2) + 4x(x-2) + 5(x-2)$

$= (x-2)(2x^2+4x+5)$

So, now finally the required limit is

$$\lim_{x\to 2} \frac{(x-2)(x^2-x-4)}{(x-2)(2x^2+4x+5)} = \lim_{x\to 2} \frac{(x^2-x-4)}{(2x^2+4x+5)} = \frac{2^2-2-4}{2(2)^2+4(2)+5} = \frac{-2}{21}$$

Ex-3. Find $\lim_{x\to 3} \dfrac{x^3-x^2-3x-9}{x^2-8x+15}$

Sol^n:- Here, the given function is a rational function whose limit is evaluated as

Observe that $\lim\limits_{x\to 3} \dfrac{x^3 - x^2 - 3x - 9}{x^2 - 8x + 15} = \dfrac{\lim\limits_{x\to 3}(x^3 - x^2 - 3x - 9)}{\lim\limits_{x\to 3}(x^2 - 8x + 15)}$

But, we can see that

$\lim\limits_{x\to 3}(x^3 - x^2 - 3x - 9) = 3^3 - (3)^2 - 3(3) - 9 = 27 - 9 - 9 - 9 = 0$

$\lim\limits_{x\to 3}(x^2 - 8x + 15) = (3)^2 - 8(3) + 15 = 9 - 24 + 15 = 0$

Next, we can factorize the polynomials $x^3 - x^2 - 3x - 9, x^2 - 8x + 15$ as follows

$x^3 - x^2 - 3x - 9$

$= (x^3 - \underline{3x^2 + 2x^2} - \underline{6x + 3x} - 9)$

$= x^2\underline{(x-3)} + 2x\ \underline{(x-3)} + 3\ \underline{(x-3)} = (x-3)(x^2 + 2x + 3)$

And similarly,

$x^2 - 8x + 15$

$= x^2 - 3x - 5x + 15$

$= x(x-3) - 5(x-3)$

$= (x-3)(x-5)$

So, now finally the required limit is

$$\lim_{x\to 3} \frac{(x-3)(x^2+2x+3)}{(x-3)(x-5)} = \lim_{x\to 3} \frac{(x^2+2x+3)}{(x-5)} = \frac{3^2 + 2(3) + 3}{3-5} = \frac{18}{-2} = -9$$

Ex-4. Find $\lim\limits_{x\to 1} \dfrac{\sqrt{x+7} - \sqrt{3x+5}}{\sqrt{3x+5} - \sqrt{5x+3}}$

Sol^n:- Here, the given limit

$$\lim_{x\to 1} \frac{\sqrt{x+7} - \sqrt{3x+5}}{\sqrt{3x+5} - \sqrt{5x+3}}$$

$$= \lim_{x\to 1} \frac{\sqrt{x+7} - \sqrt{3x+5}}{\sqrt{3x+5} - \sqrt{5x+3}} \times \frac{(\sqrt{x+7} + \sqrt{3x+5})}{(\sqrt{x+7} + \sqrt{3x+5})} \times \frac{(\sqrt{3x+5} + \sqrt{5x+3})}{(\sqrt{3x+5} + \sqrt{5x+3})}$$

[Multiplying denominator and numerator by conjugate surds]

$$= \lim_{x\to 1} \frac{x + 7 - (3x+5)}{3x + 5 - (5x+3)} \times \frac{\sqrt{3x+5} + \sqrt{5x+3}}{\sqrt{x+7} + \sqrt{3x+5}}$$

$$= \lim_{x\to 1} \frac{x + 7 - (3x+5)}{3x + 5 - (5x+3)} \times \lim_{x\to 1} \frac{\sqrt{3x+5} + \sqrt{5x+3}}{\sqrt{x+7} + \sqrt{3x+5}}$$

$$= \lim_{x\to 1} \frac{-2x+2}{-2x+2} \times \lim_{x\to 1} \frac{\sqrt{3x+5}+\sqrt{5x+3}}{\sqrt{x+7}+\sqrt{3x+5}}$$

$$= \lim_{x\to 1} 1 \times \frac{\sqrt{3+5}+\sqrt{5+3}}{\sqrt{1+7}+\sqrt{3+5}} = \frac{2\sqrt{8}}{2\sqrt{8}} = 1$$

Ex-5. Find $\lim_{x\to 3} \dfrac{\sqrt{x+1}-\sqrt{2x-2}}{\sqrt{3x+7}-\sqrt{5x+1}}$

Sol^n:- Here, the given limit

$$\lim_{x\to 3} \frac{\sqrt{x+1}-\sqrt{2x-2}}{\sqrt{3x+7}-\sqrt{5x+1}}$$

$$= \lim_{x\to 3} \frac{\sqrt{x+1}-\sqrt{2x-2}}{\sqrt{3x+7}-\sqrt{5x+1}} \times \frac{(\sqrt{x+1}+\sqrt{2x-2})}{(\sqrt{x+1}+\sqrt{2x-2})} \times \frac{(\sqrt{3x+7}+\sqrt{5x+1})}{(\sqrt{3x+7}+\sqrt{5x+1})}$$

[Multiplying denominator and numerator by conjugate surds]

$$= \lim_{x\to 3} \frac{x+1-(2x-2)}{3x+7-(5x+1)} \times \frac{\sqrt{3x+7}+\sqrt{5x+1}}{\sqrt{x+1}+\sqrt{2x-2}}$$

$$= \lim_{x\to 3} \frac{x+1-(2x-2)}{3x+7-(5x+1)} \times \lim_{x\to 3} \frac{\sqrt{3x+7}+\sqrt{5x+1}}{\sqrt{x+1}+\sqrt{2x-2}}$$

$$= \lim_{x\to 3} \frac{-x+3}{-2x+6} \times \lim_{x\to 3} \frac{\sqrt{3x+7}+\sqrt{5x+1}}{\sqrt{x+1}+\sqrt{2x-2}}$$

$$= \lim_{x\to 3} \frac{1}{2} \times \lim_{x\to 3} \frac{-x+3}{-x+3} \times \lim_{x\to 3} \frac{\sqrt{3x+7}+\sqrt{5x+1}}{\sqrt{x+1}+\sqrt{2x-2}}$$

$$= \frac{1}{2} \times 1 \times \frac{\sqrt{3(3)+7}+\sqrt{5(3)+1}}{\sqrt{3+1}+\sqrt{2(3)-2}}$$

$$= \frac{1}{2} \frac{\sqrt{16}+\sqrt{9}}{\sqrt{4}+\sqrt{4}}$$

$$= \frac{1}{2} \cdot \frac{4+3}{2+2} = \frac{7}{8}$$

Ex-6. Find $\lim_{x\to -1} \dfrac{\sqrt{8-x}-\sqrt{7-2x}}{x^3+1}$

Sol^n:- Here, the given limit

$$\lim_{x\to -1} \frac{\sqrt{8-x}-\sqrt{7-2x}}{x^3+1}$$

$$= \lim_{x\to -1} \frac{\sqrt{8-x}-\sqrt{7-2x}}{x^3+1}$$

[Multiplying denominator and numerator by conjugate surds]

$$= \lim_{x\to -1} \frac{8-x-(7-2x)}{x^3+1} \times \frac{1}{\sqrt{8-x}+\sqrt{7-2x}}$$

$$= \lim_{x\to -1} \frac{(x+1)}{x^3+1} \times \lim_{x\to -1} \frac{1}{\sqrt{8-x}+\sqrt{7-2x}}$$

$$= \lim_{x\to -1} \frac{(x+1)}{(x+1)(x^2-x+1)} \times \frac{1}{\sqrt{8+1}+\sqrt{7+2}}$$

$$= \lim_{x\to -1} \frac{1}{(x^2-x+1)} \times \frac{1}{6}$$

$$= \frac{1}{(1+1+1)} \times \frac{1}{6} = \frac{1}{18}$$

Ex-7. Find $\lim_{x\to 2} \frac{x^3-x^2-x-2}{x^2-6x+8}$

Sol^n:- Here, the given function is a rational function whose limit is evaluated as

Observe that $\lim_{x\to 2} \frac{x^3-x^2-x-2}{x^2-6x+8} = \frac{\lim_{x\to 2}(x^3-x^2-x-2)}{\lim_{x\to 2}(x^2-6x+8)}$

But, we can see that

$\lim_{x\to 2}(x^3-x^2-x-2) = 2^3-(2)^2-(2)-2 = 8-4-2-2 = 0$

$\lim_{x\to 2}(x^2-6x+8) = (2)^2-6(2)+8 = 4-12+8 = 0$

Next, we can factorize the polynomials x^3-x^2-x-2, x^2-6x+8 as follows

x^3-x^2-x-2

$= (x^3 - \underline{2x^2+x^2} - \underline{2x+x} - 2)$

$= x^2\underline{(x-2)} + x\ \underline{(x-2)} + 1\ \underline{(x-2)} = (x-2)(x^2+x+1)$

And similarly,

x^2-6x+8

$= x^2-2x-4x+8$

$= (x-2)(x-4)$

So, now finally the required limit is

$$\lim_{x\to 2} \frac{(x-2)(x^2+x+1)}{(x-2)(x-4)} = \lim_{x\to 2} \frac{(x^2+x+1)}{(x-4)} = \frac{2^2+2+1}{2-4} = \frac{7}{-2}$$

Ex-8. Find $\lim\limits_{x\to 1} \dfrac{x^2-2x+1}{x^3-3x^2+7x-5}$

Sol^n:- Here, the given function is a rational function whose limit is evaluated as

Observe that $\lim\limits_{x\to 1} \dfrac{x^2-2x+1}{x^3-3x^2+7x-5} = \dfrac{\lim\limits_{x\to 1}(x^2-2x+1)}{\lim\limits_{x\to 1}(x^3-3x^2+7x-5)}$

But, we can see that

$\lim\limits_{x\to 1}(x^2-2x+1) = 0$

$\lim\limits_{x\to 1}(x^3-3x^2+7x-5) = 0$

Next, we can factorize the polynomials x^2-2x+1, x^3-3x^2+7x-5 as follows

x^2-2x+1

$= (x-1)^2$

And similarly,

x^3-3x^2+7x-5

$= x^3-x^2-2x^2+2x+5x-5 = 0$

$= (x-1)(x^2-2x+5)$

So, now finally the required limit is

$\lim\limits_{x\to 1} \dfrac{(x-1)^2}{(x-1)(x^2-2x+5)} = \lim\limits_{x\to 1} \dfrac{(x-1)}{(x^2-2x+5)}$

$= \lim\limits_{x\to 1} \dfrac{(x-1)}{(x^2-2x+5)} = \dfrac{2^2+2+1}{2-4} = \dfrac{7}{-2}$

EX.9 Evaluate $\lim\limits_{x\to 0} \dfrac{\sin 2x}{3x}$

Sol^n:- Here, we use the above mentioned result, $\lim\limits_{x\to 0} \dfrac{\sin 2x}{3x} = \lim\limits_{x\to 0} \dfrac{\sin 2x}{2x} \cdot \dfrac{2}{3}$

$= \lim\limits_{2x\to 0} \dfrac{\sin 2x}{2x} \cdot \lim\limits_{x\to 0} \dfrac{2}{3}$

$= \dfrac{2}{3}$

EX.10 Evaluate $\lim\limits_{x\to \pi} \dfrac{\sin x}{\pi - x}$

Sol^n:- Here, we use the above mentioned result, $\lim\limits_{x\to \pi} \dfrac{\sin x}{\pi - x} = \lim\limits_{x\to \pi} \dfrac{\sin x}{\pi - x}$

Taking $y = x - \pi$ so as $x \to \pi \Rightarrow y \to 0$, hence above limit is transformed to

$$= \lim_{y\to 0} \frac{\sin(y+\pi)}{y} = \lim_{y\to 0} \frac{\sin y\cos\pi + \sin\pi\cos y}{y} = \lim_{y\to 0} \frac{-\sin y}{y} = -\lim_{y\to 0} \frac{\sin y}{y} = -1$$

EX.11 Evaluate $\lim\limits_{x\to 0} \dfrac{3\sin x - \sin 3x}{x^3}$

Sol^n:- Here, we use the above mentioned result(5),

$$\lim_{x\to 0} \frac{3\sin x - \sin 3x}{x^3}$$
$$= \lim_{x\to 0} \frac{4\sin^3 x}{x^3}$$
$$= 4 \cdot \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^3$$
$$= 4 \cdot \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^3$$
$$= 4 \cdot (1)^3 = 4$$

EX.12 Evaluate $\lim\limits_{x\to 0} \dfrac{\sin(a+x) + \sin(a-x) - 2\sin a}{x^2}$

Sol^n:- Here, we use the above mentioned result(5),

$$\lim_{x\to 0} \frac{\sin(a+x) + \sin(a-x) - 2\sin a}{x^2} = \lim_{x\to 0} \frac{2\sin a\cos x - 2\sin a}{x^2}$$

$$\because \underbrace{\sin(a+x) = \sin a\cos x + \cos a\sin x}$$
$$\text{and } \underbrace{\sin(a-x) = \sin a\cos x - \cos a\sin x}$$

$$= \lim_{x\to 0} \frac{2\sin a(\cos x - 1)}{x^2}$$
$$= 2\sin a \cdot \lim_{x\to 0} \frac{(\cos x - 1)}{x^2}$$
$$= 2\sin a \cdot \lim_{x\to 0} \frac{-(2\sin^2(x/2))}{x^2}$$
$$= -4\sin a \cdot \lim_{x\to 0} \frac{(\sin(x/2))^2}{x^2}$$
$$= -4\sin a \cdot \lim_{x\to 0} \frac{(\sin(x/2))^2}{4(x/2)^2}$$
$$= (-4/4)\sin a \cdot \lim_{x\to 0} \frac{(\sin(x/2))^2}{(x/2)^2}$$
$$= -\sin a$$

Ex.13 Evaluate $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$

Ex.14 Evaluate $\lim\limits_{x\to 1} \dfrac{1 + \cos\pi x}{\tan^2 \pi x}$

Ex.15 Obtain $\dfrac{d}{dx}x^3$ by definition.

Sol^n:- Here $f(x)=x^3$, $\dfrac{d}{dx}x^3=\lim\limits_{h\to 0}\dfrac{(x+h)^3-x^3}{h}=\lim\limits_{h\to 0}\dfrac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$
$=\lim\limits_{h\to 0}\dfrac{3x^2h+3xh^2+h^3}{h}=\lim\limits_{h\to 0}3x^2+3xh+h^2=3x^2$

Ex.16 Obtain $\dfrac{d}{dx}\dfrac{1}{2x+3}$ by definition.

Sol^n: Here, $\dfrac{d}{dx}\dfrac{1}{2x+3}=\lim\limits_{t\to x}\dfrac{f(t)-f(x)}{t-x}$

$$=\lim_{t\to x}\frac{(2t+3)^{-1}-(2x+3)^{-1}}{t-x}$$
$$=\lim_{t\to x}\frac{(2x+3)-(2t+3)/(2x+3)(2t+3)}{t-x}$$
$$=\lim_{t\to x}\frac{-2(t-x)}{(t-x)(2x+3)(2t+3)}$$
$$=\lim_{t\to x}\frac{-2}{(2x+3)(2t+3)}$$
$$=\lim_{t\to x}\frac{-2}{(2x+3)(2x+3)}$$
$$=\lim_{t\to x}\frac{-2}{(2x+3)^2}$$

UNIT-2 : Integration

"The concept of integration comes from the summation in fact, it is an infinite summation in limiting situation. It is the inverse operation of Differentiation."

Topics to be covered:

2.1 Integration: Definition

2.2 Properties of Integration

2.3 Some standard Formulas of Integration.

2.4 Simple Examples of Integration

2.5 Method of Substitution for Integration(Trigonometric Substitution)

2.6 Integration by Parts Method

2.1 *Integration:*

Definition:

If a function $g(x)$ is differentiable and if $\dfrac{d}{dx}g(x) = f(x)$, then $g(x)$ is called *integral* or *primitive* or *antiderivative* of $f(x)$ and it is denoted by $\int\ f(x)\ dx$

2.2 *Properties of Integration*

(a) $\frac{d}{dx}\left(\int f(x)\ dx\right) = f(x)$

e.g., $\frac{d}{dx}\left(\int \sin x\ dx\right) = \sin x$

(b) $\int f(x) + g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx$

e.g., $\int(\log x + \sin x)\ dx = \int \log x\ dx + \int \sin x\ dx$

(c) $\int k\ f(x)\ dx = k\int\ f(x)\ dx$, where k is a constant.

e.g.,$\int 2\ x^2\ dx = 2\int x^2\ dx$

2.3 *Some Standard Formulas of Integration*

No	Integrand f(x)	$\int$ f(x) dx
1.	x^n	$\dfrac{x^{n+1}}{n+1}+c$
2.	$\cos x$	$\sin x$
3.	$\sin x$	$-\cos x$
4.	$\sec^2 x$	$\tan x$
5.	$cosec^2 x$	$-\cot x$
6.	$\sec x \tan x$	$\sec x$
7.	$cosecx \cot x$	$-cosecx$
8.	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
9.	$\dfrac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
10.	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
11.	$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$
12.	e^x	e^x
13.	$\dfrac{1}{\sqrt{x^2\pm a^2}}$	$\log(x+\sqrt{x^2\pm a^2})$
14.	$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\log\left\|\dfrac{x-a}{x+a}\right\|$
15.	$\dfrac{1}{a^2-x^2}$	$-\dfrac{1}{2a}\log\left\|\dfrac{x-a}{x+a}\right\|$
16.	$\tan x$	$\log\|\sec x\|$
17.	$\cot x$	$\log\|\sin x\|$
18.	$cosecx$	$\log\left\|\tan\frac{x}{2}\right\|$
19.	$\sec x$	$\log\|\sec x+\tan x\|$
20.	a^x	$\dfrac{a^x}{\log_e a}$
21.	$\dfrac{f'(x)}{f(x)}$	$\log\|f(x)\|$
22.	$[f(x)]^n\cdot f'(x)$	$\dfrac{[f(x)]^{n+1}}{n+1}$

EXERCISE: Evaluate the following integration.

Ex.1. $\int (x^{3/2} - 3 \cdot 5^x - \frac{1}{x})\ dx \quad , x > 0$

Sol^n : $-$ Here we have to find $\int (x^{3/2} - 3 \cdot 5^x - \frac{1}{x})\ dx$

$$= \int x^{3/2}\ dx - \int 3 \cdot 5^x\ dx - \int \frac{1}{x}\ dx$$

$$= \frac{x^{(3/2)+1}}{(3/2)+1} - 3 \int 5^x\ dx - \log x$$

$$= \frac{x^{(3/2)+1}}{(3/2)+1} - 3\frac{5^x}{\log 5} - \log x$$

$$= \frac{x^{5/2}}{5/2} - 3\frac{5^x}{\log 5} - \log x$$

$$= \frac{2x^{5/2}}{5} - 3\frac{5^x}{\log 5} - \log x$$

EX.2 $\int \dfrac{x^3 + 3x^2 + 4}{\sqrt{x}}\ dx \quad , x > 0$

Sol^n : $-$ Here, we want to find $\int \dfrac{x^3 + 3x^2 + 4}{\sqrt{x}}\ dx$

$$= \int x^{-1/2}[x^3 + 3x^2 + 4]\ dx$$

$$= \int x^{-1/2}[x^3] + x^{-1/2}[3x^2] + x^{-1/2}[4]\ dx$$

$$= \int x^{-1/2}[x^3]\ dx + \int x^{-1/2}[3x^2]\ dx + \int x^{-1/2}[4]\ dx$$

$$= \int x^{(-1/2)+3}\ dx + \int 3x^{(-1/2)+2}\ dx + \int 4x^{-1/2}\ dx$$

$$= \int x^{5/2}\ dx + \int 3x^{3/2}\ dx + \int 4x^{-1/2}\ dx$$

$$= \frac{x^{(5/2)+1}}{(5/2)+1} + 3\frac{x^{(3/2)+1}}{(3/2)+1} + 4\frac{x^{(-1/2)+1}}{(-1/2)+1}$$

$$= \frac{x^{7/2}}{7/2} + 3\frac{x^{5/2}}{5/2} + 4\frac{x^{1/2}}{1/2}$$

$$= \frac{2x^{7/2}}{7} + \frac{6x^{5/2}}{5} + 8x^{1/2}$$

Ex.3. $\int(\sin x+e^x+4^x+x^4)\ dx$

Sol^n : $-$ Here we want to prove that $\int(\sin x+e^x+4^x+x^4)\ dx$

$$=\int\sin x\ dx+\int e^x\ dx+\int 4^x\ dx+\int x^4\ dx$$

$$=-\cos x+e^x+\frac{4^x}{\log 4}+\frac{x^5}{5}$$

Ex.4. $\int\frac{1}{4x^2+9}\ dx$

Sol^n : $-$ Here we have to prove that $\int\frac{1}{4x^2+9}\ dx$

$$=\int\frac{1}{4(x^2+\frac{9}{4})}\ dx$$
$$=\frac{1}{4}\int\frac{1}{(x^2+\frac{9}{4})}\ dx$$
$$=\frac{1}{4}\int\frac{1}{(x^2+\frac{9}{4})}\ dx$$

Using the formula

$$\int\frac{1}{x^2+a^2}\ dx=\frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$=\frac{1}{4}\frac{1}{3/2}\tan^{-1}\frac{x}{3/2}$$

$$=\frac{1}{4}\cdot\frac{2}{3}\tan^{-1}\frac{2x}{3}$$

$$=\frac{1}{6}\tan^{-1}\frac{2x}{3}$$

Ex.5. $\int\frac{\cos x}{\cos x-1}\ dx$

Sol^n : $-$ Here we have to prove that $\int\frac{\cos x}{\cos x-1}\ dx$

$$=\int\left(\frac{\cos x}{\cos x-1}\times\frac{\cos x+1}{\cos x+1}\right)\ dx$$

$$=\int\left(\frac{\cos x(\cos x+1)}{\cos^2 x-1}\right)\ dx$$

$$=\int\frac{\cos x(\cos x+1)}{\sin^2 x}\ dx$$

$$=\int\frac{\cos^2 x+\cos x}{\sin^2 x}\ dx$$

$$=\int\frac{\cos^2 x}{\sin^2 x}\ dx+\int\frac{\cos x}{\sin^2 x}\ dx$$

$$=\int\cot^2 x\ \ dx+\int\cot x\cdot cosecx\ \ dx$$

$$= \int 1 - cosec^2 x \;\; dx + \int \cot x \cdot cosecx \;\; dx$$

$$= \int 1 \; dx - \int cosec^2 x \; dx \; + \int \cot x \cdot cosecx \;\; dx$$

$$= x - (-\cot x) - cosecx$$

$$= x + \cot x - cosecx$$

EXERCISE:. Evaluate the following integration.

EX.1. $\int x\sqrt{x+2}\; dx, \quad x > -2$

Sol^n : $-$ Here we want to find $\int x\sqrt{x+2}\; dx, \quad x > -2$

$I = \int x\sqrt{x+2}\; dx$

Here, the above integral contains the term $\sqrt{x+2}$, so let substitution $\sqrt{x+2} = t \Rightarrow x+2 = t^2 \Rightarrow x = t^2 - 2$

$\Rightarrow \dfrac{d}{dt}(x+2) = \dfrac{d}{dt}(t^2) \Rightarrow \dfrac{dx}{dt} = 2t \Rightarrow dx = 2tdt$

Using this into I we get,

$$I = \int (t^2 - 2)t(2tdt)$$

$$= \int 2(t^2 - 2)t^2 dt$$

$$= \int (2t^4 - 2t^2)dt$$

$$= \int 2t^4 \; dt - \int 2t^2 dt$$

$$= 2\frac{t^5}{5} - 2\frac{t^3}{3}$$

But, we have taken $t = \sqrt{x+2}$, so we have to substitute it back

$$I = \frac{2}{5}(\sqrt{x+2})^5 - \frac{2}{3}(\sqrt{x+2})^3$$

EX.2. $\int \dfrac{x-1}{\sqrt{x+4}}\; dx, x > -4$

Sol^n : $-$ Here we have to find $I = \int \dfrac{x-1}{\sqrt{x+4}}\; dx$

Here the integrand contains the term $\sqrt{x+4}$, so take $\sqrt{x+4} = t$

$\Rightarrow x + 4 = t^2 \;\; \Rightarrow x = t^2 - 4 \;\; \Rightarrow \dfrac{d}{dt}(x) = \dfrac{d}{dt}(t^2 - 4) \;\; \Rightarrow \dfrac{dx}{dt} = 2t \;\; \Rightarrow dx = 2tdt$

Using this into the integral I, we get

$$I = \int \frac{(t^2-4)-1}{t}\ (2t\ dt)$$
$$= \int 2(t^2-5)\ dt$$
$$= \int 2t^2 - 10\ dt$$
$$= \int 2t^2\ dt - \int 10\ dt$$
$$= \frac{2t^3}{3} - 10t$$

But we have taken $t = \sqrt{x+4}$, so we have to substitute it back.

$$I = \frac{2(\sqrt{x+4})^3}{3} - 10\sqrt{x+4}$$

EX.3. $\int \frac{1-\tan x}{1+\tan x}\ dx$

Sol^n : $-$ Here we have to find $I = \int \frac{1-\tan x}{1+\tan x}\ dx$

$$I = \int \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\ dx$$
$$= \int \frac{(\cos x - \sin x)/\cos x}{(\cos x + \sin x)/\cos x}\ dx$$
$$= \int \frac{(\cos x - \sin x)}{(\cos x + \sin x)}\ dx$$

Let $t = \cos x + \sin x \Rightarrow dt = (-\sin x + \cos x)\ dt\ \Rightarrow dt = (\cos x - \sin x)dx$

Using this into I, we get

$$I = \int \frac{dt}{t}$$
$$I = \log t\ \Rightarrow\ I = \log(\cos x + \sin x)$$

EX.4. $\int \frac{e^{2x}+1}{e^{2x}-1}\ dx$

Sol^n : $-$ Here we have to find $I = \int \frac{e^{2x}+1}{e^{2x}-1}\ dx$

$$= \int \frac{e^{2x}+1}{e^{2x}-1}\ dx$$
$$= \int \frac{e^{-x}(e^{2x}+1)}{e^{-x}(e^{2x}-1)}\ dx$$
$$= \int \frac{(e^{x}+e^{-x})}{(e^{x}-e^{-x})}\ dx$$
$$= \int \frac{(e^{x}+e^{-x})}{(e^{x}-e^{-x})}\ dx$$

Let $e^x - e^{-x} = t \Rightarrow (e^x - e^{-x}(-1))dx = dt \Rightarrow (e^x + e^{-x})dx = dt$

Using this into I, we get,

$$I = \int \frac{dt}{t} = \log t = \log(e^x - e^{-x})$$

EX.5. $\int x^{4x} \cdot (1 + \log x)\ dx \quad , x > 0$

Sol^n : $-$ Here we have to find $\int x^{4x} \cdot (1 + \log x)\ dx$

$$I = \int x^{4x} \cdot (1 + \log x)\ dx$$

$$= \int x^{3x} \cdot x^x\ (1 + \log x)\ dx$$

$$= \int x^{3x} \cdot [x^x\ (1 + \log x)]\ dx$$

Here, we can see that,

For $t = x^x$

Next to find $\dfrac{dt}{dx}$ we have to use log as follows,

$t = x^x \Rightarrow \log t = \log x^x \Rightarrow \log t = x \log x$

Now differentiating with respect to x, we get,

$\dfrac{1}{t} \cdot \dfrac{dt}{dx} = \dfrac{d}{dx}(x \log x)$

$\Rightarrow \dfrac{dt}{dx} = t\left[\dfrac{dx}{dx}(\log x) + x\dfrac{d}{dx}(\log x)\right]$

$\Rightarrow \dfrac{dt}{dx} = x^x\left[\log x + x\dfrac{1}{x}\right]$

$\Rightarrow \dfrac{dt}{dx} = x^x[\log x + 1]$

$\Rightarrow dt = x^x[\log x + 1]\ dx$

Now,

$$I = \int (x^x)^3[x^x(\log x + 1)]\ dx$$

using the above substitutions into I, we get,

$$I = \int t^3\ dt$$

$$I = \frac{t^4}{4} = \frac{(x^x)^4}{4} = \frac{x^{4x}}{4}$$

EX.6. $\int \dfrac{\sin x}{\sin 3x}\ dx$

Sol^n : $-$ Here we have to find $\int \dfrac{\sin x}{\sin 3x}\, dx$

$$I = \int \frac{\sin x}{\sin 3x}\, dx$$

We know that $\sin 3x = 3\sin x - 4\sin^3 x$, so using this into above integral we get,

$$I = \int \frac{\sin x}{3\sin x - 4\sin^3 x}\, dx$$

Dividing numerator and denominator both by $\sin^3 \theta$, we get,

$$\begin{aligned} I &= \int \frac{(\sin x)/(\sin^3 x)}{(3\sin x - 4\sin^3 x)/\sin^3 x}\, dx \\ &= \int \frac{cosec^2 x}{3cosec^2 x - 4}\, dx \end{aligned}$$

Now, we take the substitution $\cot x = t \;\Rightarrow -cosec^2 x\; dx = dt$ into I,we get

$$\begin{aligned} I = \int \frac{cosec^2 x}{3(1+\cot^2 x) - 4}\, dx &= \int \frac{-\, dt}{3(1+t^2) - 4} \\ &= -\int \frac{dt}{3 + 3t^2 - 4} \\ &= -\int \frac{dt}{3t^2 - 1} \end{aligned}$$

Now, by the formula

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right|$$

we get,

$$\begin{aligned} I &= -\int \frac{dt}{3\left(t^2 - \frac{1}{3}\right)} \\ &= -\frac{1}{3}\int \frac{dt}{\left(t^2 - \frac{1}{3}\right)} \\ &= -\frac{1}{3}\int \frac{dt}{\left(t^2 - (\frac{1}{\sqrt{3}})^2\right)} \\ &= -\frac{1}{3}\frac{1}{2(1/\sqrt{3})} \log \left| \frac{t - (1/\sqrt{3})}{t + (1/\sqrt{3})} \right| \\ &= -\frac{\sqrt{3}}{6} \log \left| \frac{\sqrt{3}t - 1}{\sqrt{3}t + 1} \right| \\ &= -\frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3}t - 1}{\sqrt{3}t + 1} \right| \end{aligned}$$

EX.7. $\int \dfrac{(\log x)^n}{x}\, dx, x > 0$

Sol^n : $-$ Here we have to find $\int \frac{(\log x)^n}{x}\ dx, x > 0$

$$I = \int \frac{(\log x)^n}{x}\ dx$$
$$= \int (\log x)^n \cdot \frac{1}{x}\ dx$$
$$= \int (\log x)^n \cdot \frac{d}{dx}(\log x)\ dx$$

Now, using the formula

$$\int [f(x)]^n f'(x)\ dx = \frac{[f(x)]^{n+1}}{n+1}$$

we get,

$$I = \frac{[\log x]^{n+1}}{n+1}$$

2.5 *Method of Trigonometric substitution in Integration*

Sometimes using proper trigonometric substitutions we can transform given integral into a simple form whose integration can be easily obtained.

(a) *List of Some Trigonometric Identities Useful in Substitution*

No.	Trigonometric Identities
1.	$\sin^2\theta + \cos^2\theta = 1$
2.	$\sec^2\theta - \tan^2\theta = 1$
3.	$\cos 2\theta = \cos^2\theta - \sin^2\theta$
4.	$\sin 2\theta = 2\sin\theta\cos\theta$
5.	$\sin^2\theta = \frac{1-\cos 2\theta}{2}$
6.	$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

(b) ***List of Some Trigonometric Substitutions***

No.	Integrand Involving the term	Substitution
1.	$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
2.	$\sqrt{x^2 - a^2}$	$x = a\sec\theta$
3.	$\sqrt{x^2 + a^2}$	$x = a\tan\theta$
4.	$\sqrt{x+a}$ or $\sqrt{x-a}$	$x = a\cos 2\theta$
5.	$\sqrt{a-x}$	$x = a\sin^2\theta$
6.	$\sqrt{a+x}$	$x = a\tan^2\theta$
7.	$\sqrt{2ax - x^2} = \sqrt{a^2 - (x-a)^2}$	$x - a = a\sin\theta$

EXERCISE: Evaluate the following integration.

EX.1. $\int x^2 \sqrt{a^6 - x^6}\, dx \quad (a > 0)$

Sol^n : $-$ Here we have to find $\int x^2 \sqrt{a^6 - x^6}\, dx$

$$I = \int x^2 \sqrt{a^6 - x^6}\, dx$$

Here, by observing the term $\sqrt{a^6 - x^6}$ and the above list of the substitutions we should select the substitution $x^3 = a^3 \sin\theta$

OR

Using the formula $\sin^2\theta + \cos^2\theta = 1$
that implies $\cos^2\theta = 1 - \sin^2\theta$
Comparing with $\cos^2\theta = a^6 - x^6$
$\Rightarrow 1 - \sin^2\theta \approx a^6\left(1 - \frac{x^6}{a^6}\right)$
$\Rightarrow \sin^2\theta = \frac{x^6}{a^6}$
so, the appropriate substitution is $x^3 = a^3\sin\theta$

$$\Rightarrow 3x^2\, dx = a^2\cos\theta\, d\theta \;\; \Rightarrow x^2\, dx = \frac{1}{3}a^2\cos\theta\, d\theta$$

Here,

$$I = \int \sqrt{a^6 - x^6}\, (x^2\, dx)$$

Using the above substitutions into I, we get

$$I = \int \sqrt{a^6 - a^6 \sin^2\theta} \left(\frac{1}{3} a^3 \cos\theta d\theta\right)$$
$$= \frac{1}{3} \int \sqrt{a^6}\sqrt{1 - \sin^2\theta}\ a^3 \cos\theta\ d\theta$$
$$= \frac{1}{3} \int a^3 \sqrt{\cos^2\theta}\ a^3 \cos\theta\ d\theta$$
$$= \frac{1}{3} a^3 a^3 \int \cos\theta\ \cos\theta\ d\theta$$
$$= \frac{a^6}{3} \int \cos^2\theta\ d\theta$$
$$= \frac{a^6}{3} \int \frac{1 + \cos 2\theta}{2}\ d\theta$$
$$= \frac{a^6}{3 \cdot 2} \int 1 + \cos 2\theta\ d\theta$$
$$= \frac{a^6}{6} \left[\int 1\ d\theta + \int \cos 2\theta\ d\theta\right]$$

$$I = \frac{a^6}{6}\left[\theta + \frac{\sin 2\theta}{2}\right] = \frac{a^6}{6}\left[\theta + \frac{2\sin\theta \cdot \cos\theta}{2}\right] = \frac{a^6}{6}\left[\theta + \sin\theta \cdot \cos\theta\right]$$

But, we have taken $x^3 = a^3 \sin\theta \ \Rightarrow \sin\theta = \dfrac{x^3}{a^3}$

$\Rightarrow \cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \dfrac{x^6}{a^6}} = \sqrt{\dfrac{a^6 - x^6}{a^6}} = \dfrac{1}{a^3}\sqrt{a^6 - x^6}$

And $\sin\theta = \dfrac{x^3}{a^3} \ \Rightarrow \theta = \sin^{-1}\dfrac{x^3}{a^3}$

Hence,

$$I = \frac{a^6}{6}\left[\sin^{-1}\frac{x^3}{a^3} + \frac{x^3}{a^3} \cdot \frac{1}{a^3}\sqrt{a^6 - x^6}\right]$$
$$I = \frac{a^6}{6}\left[\sin^{-1}\frac{x^3}{a^3} + \frac{x^3}{a^6}\sqrt{a^6 - x^6}\right]$$

EX.2. $\int \dfrac{1}{\sqrt{2ax - x^2}}\ dx \qquad (0 < x < 2a)$

Sol^n : $-$ Here we have to find $\int \dfrac{1}{\sqrt{2ax - x^2}}\ dx$

$$I = \int \frac{1}{\sqrt{2ax - x^2}}\ dx$$

Here, we the given integral contains

$\sqrt{2ax - x^2} = \sqrt{a^2 - a^2 + 2ax - x^2} = \sqrt{a^2 - (a^2 - 2ax + x^2)}$

$= \sqrt{a^2 - (x^2 - 2ax + a^2)} = \sqrt{a^2 - (x - a)^2}$

So,

$$I = \int \frac{1}{\sqrt{a^2 - (x-a)^2}}\ dx$$

Next, using the substitution $x - a = a\sin\theta$

$\Rightarrow\ dx = a\cos\theta\ d\theta$

Hence, we get,

$$\begin{aligned} I &= \int \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}} a\cos\theta\ d\theta \\ &= \int \frac{1}{a\sqrt{1-\sin^2\theta}} a\cos\theta\ d\theta \\ &= \int \frac{1}{a\cos\theta} a\cos\theta\ d\theta \\ &= \int d\theta = \theta \end{aligned}$$

But we have taken $x - a = a\sin\theta\ \Rightarrow \sin\theta = \dfrac{x-a}{a}\ \Rightarrow \theta = \sin^{-1}\left(\dfrac{x-a}{a}\right)$

So,

$$I = \sin^{-1}\left(\frac{x-a}{a}\right)$$

EX.3. $\int \dfrac{\sqrt{3-x}}{x}\ dx \quad x \in (0,3)$

Sol^n : $-$ Here we have to find $\int \dfrac{\sqrt{3-x}}{x}\ dx$

$$I = \int \frac{\sqrt{3-x}}{x}\ dx$$

Here, observing the given integral it contains the term $\sqrt{3-x}$

Using the formula $\cos^2\theta = 1 - \sin^2\theta$ comparing with $3 - x = 3\left(1 - \dfrac{x}{3}\right)$

We get,the proper substitution $\dfrac{x}{3} = \sin^2\theta\ \Rightarrow x = 3\sin^2\theta$

$\Rightarrow dx = 3(2\sin\theta)\dfrac{d}{d\theta}(\sin\theta)\ d\theta\ \Rightarrow dx = 6\sin\theta\cdot\cos\theta\ \ d\theta.$

Using the above substitution into I, we get,

$$\begin{aligned} I &= \int \frac{\sqrt{3 - 3\sin^2\theta}}{3\sin^2\theta}\ (6\sin\theta\cdot\cos\theta\ d\theta) \\ &= \int \frac{\sqrt{3(1-\sin^2\theta)}}{3\sin\theta}\ 6\cos\theta\ d\theta \\ &= \frac{6\sqrt{3}}{3}\int \frac{\sqrt{\cos^2\theta}}{\sin\theta}\ \cos\theta\ d\theta \end{aligned}$$

$$= 2\sqrt{3} \int \frac{\cos\theta}{\sin\theta} \cos\theta \; d\theta$$

$$= 2\sqrt{3} \int \frac{\cos^2\theta}{\sin\theta} \; d\theta$$

$$= 2\sqrt{3} \int \frac{1-\sin^2\theta}{\sin\theta} \; d\theta$$

$$= 2\sqrt{3} \left[\int \frac{1}{\sin\theta} \; d\theta - \int \frac{\sin^2\theta}{\sin\theta} \; d\theta \right]$$

$$= 2\sqrt{3} \left[\int cosec\theta \; d\theta - \int \sin\theta \; d\theta \right]$$

$$= 2\sqrt{3} \left[\log\left|\tan\frac{\theta}{2}\right| - (-\cos\theta) \right]$$

$$= 2\sqrt{3} \left[\log\left|\tan\frac{\theta}{2}\right| + \cos\theta \right]$$

But, we have taken $x = 3\sin^2\theta \;\Rightarrow \frac{x}{3} = \sin^2\theta \;\Rightarrow \sqrt{\frac{x}{3}} = \sin\theta$

$\Rightarrow \theta = \sin^{-1}\sqrt{\frac{x}{3}}$

So, using this, we get,

$I = 2\sqrt{3} \left[\log\left|\tan\frac{\sin^{-1}\sqrt{x/3}}{2}\right| + \cos\left(\sin^{-1}\sqrt{\frac{x}{3}}\right) \right]$

2.6 *Integration By Parts Method*

$\gg$ **Rule of Integration by Parts:**

If

(i) the two functions $u(x)$ and $v(x)$ are differentiable functions

(ii) u', v' are continuous then

$$\boxed{\int u \; v \; dx = u \int \; v \; dx - \int (\tfrac{d}{dx}u) \cdot (\int \; v \; dx) \; dx}$$

$\gg$ Note:-

1. While using the above formula we have to be careful about the choice of u and v, so that the integration should be easier.

2. For the choice of the function "u" we follow the priority order : **LIATE** *(L: Logarithmic function, I: Inverse function, A: Algebraic function, T: Trigonometric function, E: Exponential function.)*

   $Log(\log x, \log(x+1), ...)$
   $\gg Inverse(\tan^{-1}, \cos^{-1}, \sin^{-1}, ..)$
   $\gg Algebraic(1, x, x^2, 1+x, 2+7x, x^3, ...)$

$\gg$ *Trigonometric*(sin, cos, tan, ...)
$\gg$ *Exponential*($e^x, e^{x+1}, ...$)

3. Sometimes we use the method of integration by parts to get integrals of one function only. In that case we take v=1.
   Thus we will obtain integrals of $\log x, \sin^{-1} x, \tan^{-1} x, ...$ using this method.

$\gg$EXERCISE:- Evaluate the following integration.

EX.1. $\int xe^x \, dx$

Sol^n : $-$ Here we have to find $\int xe^x \, dx$

We use integration by parts method.

$$I = \int xe^x \, dx$$

Here, the above integrand is the product of two functions x and e^x, so we use the integration by parts method.

By the priority order LIATE,
here A: Algebraic function x, comes before the E: Exponential function e^x

So, we have to select $u = x$ and $v = e^x$
Using them into the formula

$$\int u \cdot v \, dx = u \left(\int v \, dx\right) - \int \left(\frac{d}{dx}(u)\right)\left(\int v \, dx\right) \, dx$$

We get,

$$\int x \cdot e^x \, dx = x \left(\int e^x \, dx\right) - \int \left(\frac{d}{dx}(x)\right)\left(\int e^x \, dx\right) \, dx$$

$$= x \, e^x - \int (1) \cdot (e^x) \, dx$$

$$= x \, e^x - \int e^x \, dx$$

$$I = x \, e^x - e^x = e^x(x-1)$$

EX.2. $\int x \cos x \, dx$

Sol^n : $-$ Here we have to find $\int x \cos x \, dx$

$$Let \ I = \int x \cos x \, dx$$

Here, the above integrand is the product of two functions x and $\cos x$, so we use the integration by parts method.
By the priority order LIATE,

here A: Algebraic function x, comes before the T: Trigonometric function $\cos x$

So, we have to select $u = x$ and $v = \cos x$
And using them into the formula,

$$\int u \cdot v \ dx = u \ \left(\int v \ dx\right) - \int \left(\frac{d}{dx}(u)\right)\left(\int v \ dx\right) \ dx$$

we get,

$$\int x \cdot \cos x \ dx = x \ \left(\int \cos x \ dx\right) - \int \left(\frac{d}{dx}(x)\right)\left(\int \cos x \ dx\right) \ dx$$

$$= x(\sin x) - \int (1) \cdot (\sin x) \ dx$$

$$= x(\sin x) - \int (\sin x) \ dx$$

$$= x(\sin x) - (-\cos x)$$

$$I = x \sin x + \cos x$$

EX.3. $\int x \log x \ dx$

Sol^n : $-$ Here we have to find $\int x \log x \ dx$

$$Let \ I = \int x \ \log x \ dx$$

Here, the above integrand is the product of two functions x and $\log x$, so we use the integration by parts method.

By the priority order LIATE,
here L: Logarithmic function $\log x$, comes before the A: Algebraic function x

So, we have to select $u = \log x$ and $v = x$
And using them into the formula,

$$\int u \cdot v \ dx = u \ \left(\int v \ dx\right) - \int \left(\frac{d}{dx}(u)\right)\left(\int v \ dx\right) \ dx$$

we get,

$$\int \log x \cdot x \ dx = \log x \ \left(\int x \ dx\right) - \int \left(\frac{d}{dx}(\log x)\right)\left(\int x \ dx\right) \ dx$$

$$= \log x \ \left(\frac{x^2}{2}\right) - \int \left(\frac{1}{x}\right)\left(\frac{x^2}{2}\right) \ dx$$

$$= \left(\frac{x^2}{2}\right) \log x - \int \left(\frac{x}{2}\right) \ dx$$

$$= \frac{x^2}{2} \log x - \frac{1}{2} \int x \ dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\frac{x^2}{2}$$

$$= \frac{x^2}{2}\log x - \frac{x^2}{4}$$

$$I = \frac{x^2}{4}[2\log x - 1]$$

EX.4. $\int (2+7x)\cos 6x \ dx$

Sol^n : $-$ Here we have to find $\int (2+7x)\cos 6x \ dx$

$$Let\ I = \int (2+7x)\cos 6x \ dx$$

Here, the above integrand is the product of two functions $(2+7x)$ and $\cos 6x$, so we use the integration by parts method.

By the priority order LIATE,here
A: Algebraic function $(2+7x)$, comes before the T: Trigonometric function $\cos 6x$

So, we have to select $u = (2+7x)$ and $v = \cos 6x$
And using them into the formula,

$$\int u \cdot v \ dx = u \ \left(\int v \ dx\right) - \int \left(\frac{d}{dx}(u)\right)\left(\int v \ dx\right) \ dx$$

we get,

$$\int (2+7x)\cdot \cos 6x \ dx = (2+7x) \ \left(\int \cos 6x \ dx\right) - \int \left(\frac{d}{dx}(2+7x)\right)\left(\int \cos 6x \ dx\right) \ dx$$

$$= (2+7x)\ \left(\frac{\sin 6x}{6}\right) - \int (0+7)\left(\frac{\sin 6x}{6}\right) \ dx$$

$$= \frac{1}{6}(2+7x)\sin 6x \ - \frac{7}{6}\int \sin 6x \ dx$$

$$= \frac{1}{6}(2+7x)\sin 6x \ - \frac{7}{6}\frac{(-\cos 6x)}{6}$$

$$I = \frac{1}{6}(2+7x)\sin 6x \ + \frac{7}{36}\cos 6x$$

EX.5. $\int \log x \ dx$

Sol^n : $-$ Here we have to find $\int \log x \ dx$

$$Let\ I = \int \log x \ dx$$

We don't have any formula for the integration of $\log x$

We can see the $\log x$ as
$(\log x)\cdot 1 = u \cdot v$
So,

$$I = \int (\log x) \cdot 1 \; dx$$

Here, the above integrand is the product of two functions $\log x$ and 1, so we use the integration by parts method.

By the priority order LIATE,here
L: Logarithmic function $\log x$, comes before the A: Algebraic function 1

So, we have to select $u = \log x$ and $v = 1$
And using them into the formula,

$$\int u \cdot v \; dx = u \left(\int v \; dx \right) - \int \left(\frac{d}{dx}(u) \right) \left(\int v \; dx \right) \; dx$$

we get,

$$\int \log x \cdot 1 \; dx = \log x \left(\int 1 \; dx \right) - \int \left(\frac{d}{dx}(\log x) \right) \left(\int 1 \; dx \right) \; dx$$

$$= (\log x) \; x - \int \frac{1}{x} \cdot x \; dx$$

$$= x \log x - \int 1 \; dx$$

$$I = x \log x - x$$

EXTRA EXAMPLES :

EXERCISE: Evaluate the following integration.

(1) $\int \left(\frac{x}{a} + \frac{a}{x} + x^a + a^x + ax \right) \; dx$

(2) $\int \dfrac{1}{\sqrt{2x^2+3}} \; dx$

(3) $\int \dfrac{\cos 2x}{\sin^2 2x} \; dx$

(4) $\int \dfrac{a + b\cos x}{\sin^2 x} \; dx$

(5) $\int \left(e^{a \log x} + e^{x \log a} \right) \; dx$

(6) $\int \sqrt{1 - \cos x} \; dx, \quad 0 < x < \pi$

Method of Substitution

(7) $\int \tan^3 x \; dx$

(8) $\int \dfrac{x^2}{1+x^6} \; dx$

(9) $\int \dfrac{e^x}{e^{2x}+1} \; dx$

(10) $\int \dfrac{dx}{1 - \tan x}$

(11) $\int \dfrac{dx}{x^2\sqrt{1-x^2}}$

Use Integration by parts method:

(12) $\int (\log x)^2 \ dx$

(13) $\int x^2 e^{3x} \ dx$

(14) $\int x^3 \tan^{-1} x \ dx$

(15) $\int \dfrac{x}{1-\cos x} \ dx$

UNIT - 3: US02EMTH02

(ELECTIVE MATHS, SEM. II)

Definite Integration

"We know integration (in fact antiderivative) as an inverse operator of differentiation. From Historical point of view, the concept of Integration has its origin to the problem of finding area of a plane bounded region. The definite integral was expressed as a limit of certain sum expressing the area of some region. Later on the link between apparently two different concepts of differentiation and integration was established in 17^{th} century by well-known mathematician Leibnitz. The relation is known as Fundamental Theorem of Integration."

Topics to be covered

3.1 Definite Integration: Definition
3.2 Fundamental Principle of definite integration
3.3 Working Rules of Definite integration.
3.4 Statements of some useful results about definite integration.
3.5 Application of Fundamental Principle of definite integration
3.6 Integration by Parts method for Definite Integrals.

Reference book : Gujarat State Board of School Textbook
Standard - 12 MATHEMATICS - 2,(CHAPTER - 7)

3.1 Definition: Definite integration.

The indefinite integral in a limiting situation

$$\int_a^b f(x)\ dx$$

is known as definite integral a function $f(x)$.

More precisely we can define it as follows:-

Divide the interval $[a, b]$ into n-equal parts the length $h = \dfrac{b-a}{n}$

Then consider the sum

$$S = \lim_{n\to\infty} h \sum_{i=0}^{n-1} f(a + ih)$$

is the value of the definite integral

$$\int_a^b f(x)\ dx$$

Hence,

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} h \sum_{i=0}^{n-1} f(a+ih)$$

3.2 *Fundamental principle of definite integration*

History:

THIS PRINCIPLE ESTABLISHES A RELATION BETWEEN THE PROCESS OF DIFFERENTIATION AND INTEGRATION. NEWTON AND LEIBNITZ INDEPENDENTLY OBTAINED THIS RESULT. WITH THE HELP OF THIS RESULT WE CAN OBTAIN THE DEFINITE INTEGRAL OF A FUNCTION OVER AN INTERVAL BY TAKING DIFFERENCE OF VALUES OF ITS PRIMITIVE AT THE GIVEN INTERVAL.

Statement: If a function f is continuous on $[a,b]$ and F is a differentiable function on $[a,b]$ such that $\dfrac{d}{dx}[F(x)] = f(x)$ then

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a).$$

Example: Using the fundamental principle of definite integral evaluate: $\int_0^1 (x^2+3)dx$

Sol^n: $\int_0^1 (x^2+3)dx = \int_0^1 (x^2)\ dx + \int_0^1 3\ dx = \left[\frac{x^3}{3}\right]_0^1 + 3[x]_0^1 = \frac{1}{3} + 3 = \frac{10}{3}$

3.3 *Working rules of definite integration*

⋙ $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx.$

⋙ $\int_a^a f(x)\ dx = 0.$

⋙ For $a < c < b$ then $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$

⋙ $\int_a^b [f(x)+g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx.$

⋙ $\int_a^b k\ f(x)\ dx = k \int_a^c f(x)\ dx$, where k is a constant.

3.4 *Statements of some useful results about definite integration.*

– If f(x) is even function then $\int_{-a}^a f(x)\ dx = 2\int_0^a f(x)\ dx.$

– If f(x) is odd function then $\int_{-a}^a f(x)\ dx = 0.$

– $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx.$

– $\int_a^b f(x)\ dx = \int_a^b f(a+b-x)\ dx.$

$$-\int_0^{2a} f(x)\ dx = \int_0^{a} f(x)\ dx + \int_0^{a} f(2a-x)\ dx.$$

3.5 *Application of Fundamental Principle of definite integration.*

Exercise: **Evaluate the following integrals.**

(1) $\int_0^1 \frac{2x+3}{5x^2+1}\ dx$

Sol.: Here we want to find $\int_0^1 \frac{2x+3}{5x^2+1}\ dx$

Let $I = \int_0^1 \frac{2x+3}{5x^2+1}\ dx = \int_0^1 \frac{2x}{5x^2+1}\ dx + \int_0^1 \frac{3}{5x^2+1}\ dx = I_1 + I_2$

$I_1 = \int_0^1 \frac{2x}{5x^2+1}\ dx, \qquad I_2 = \int_0^1 \frac{3}{5x^2+1}\ dx$

$$I_1 = \int_0^1 \frac{2x}{5x^2+1}\ dx$$

$$= \frac{1}{5}\int_0^1 \frac{10x}{5x^2+1}\ dx$$

$$= \frac{1}{5}\int_0^1 \frac{\frac{d}{dx}(5x^2+1)}{5x^2+1}\ dx$$

$$= \frac{1}{5}\left[\log(5x^2+1)\right]_0^1$$

$$= \frac{1}{5}\left[\log(5(1)^2+1) - \log(5(0)^2+1)\right]$$

$$= \frac{1}{5}[\log 6 - \log 1]$$

$$= \frac{1}{5}[\log 6 - 0]$$

$$I_1 = \frac{1}{5}[\log 6]$$

Next,we evaluate I_2

$$I_2 = \int_0^1 \frac{3}{5x^2+1}\ dx$$

$$= \int_0^1 \frac{3}{5(x^2+\frac{1}{5})}\ dx$$

$$= \frac{3}{5}\int_0^1 \frac{1}{x^2+\frac{1}{5}}\ dx$$

$$= \frac{3}{5}\int_0^1 \frac{1}{x^2+\frac{1}{5}}\ dx$$

By the formula: $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\ \tan^{-1}\left(\frac{x}{a}\right)$, we get,

$$I_2 = \frac{3}{5} \cdot \frac{1}{1/\sqrt{5}}\ \tan^{-1}\left(\frac{x}{1/\sqrt{5}}\right)$$
$$= \frac{3\sqrt{5}}{5}\ \tan^{-1}\left(\sqrt{5}x\right)$$
$$I_2 = \frac{3}{\sqrt{5}}\ \tan^{-1}\left(\sqrt{5}x\right)$$

So,

$$I = I_1 + I_2 = \frac{1}{5}\left[\log 6\right] + \frac{3}{\sqrt{5}}\ \tan^{-1}\left(\sqrt{5}x\right)$$

(2) $\int\limits_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(1+\cos\theta)^2}\ d\theta$

Sol.: Here we have to evaluate $\int\limits_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(1+\cos\theta)^2}\ d\theta$

$$Let\ I = \int\limits_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(1+\cos\theta)^2}\ d\theta$$
$$= \int\limits_0^{\frac{\pi}{2}} \left(\frac{\sin\theta}{1+\cos\theta}\right)^2\ d\theta$$
$$= \int\limits_0^{\frac{\pi}{2}} \left(\frac{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right)^2\ d\theta$$
$$= \int\limits_0^{\frac{\pi}{2}} \left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)^2\ d\theta$$
$$= \int\limits_0^{\pi/2} \tan^2\frac{\theta}{2}\ d\theta.$$
$$= \int\limits_0^{\pi/2} (\sec^2\frac{\theta}{2} - 1)\ d\theta.$$
$$= \int\limits_0^{\pi/2} \sec^2\frac{\theta}{2} d\theta - \int\limits_0^{\pi/2}\ d\theta.$$
$$= \left[\frac{\tan\frac{\theta}{2}}{1/2}\right]_0^{\pi/2} - [\theta]_0^{\pi/2}.$$
$$= 2\left[\tan\frac{\pi/2}{2} - \tan\frac{0}{2}\right] - \left[\frac{\pi}{2} - 0\right].$$
$$= 2\tan\frac{\pi}{4} - \frac{\pi}{2}.$$
$$= 2(1) - \frac{\pi}{2} \Rightarrow\ I = \frac{4-\pi}{2}.$$

(3) Evaluate $\int\limits_1^4 f(x)\ dx$,where $f(x) = \begin{cases} 2x+8, & 1 \le x \le 2 \\ 6x, & 2 < x \le 4 \end{cases}$

Sol.: Let,

$$I = \int\limits_1^4 f(x)\ dx$$

Here, we can see that the lower limit is 1 and upper limit is 4 and the function $f(x)$ is defined on two parts $1 \le x \le 2$ and $2 \le x \le 4$.
Using the working rule of definite integration, we have

$$I = \int_1^4\ f(x)\ dx = \int_1^2\ f(x)\ dx + \int_2^4\ f(x)\ dx$$

$$\Rightarrow\ I = \int_1^2\ (2x+8)\ dx + \int_2^4\ 6x\ dx$$

$$\Rightarrow\ I = \left[2\frac{x^2}{2} + 8x\right]_1^2 + \left[6\frac{x^2}{2}\right]_2^4$$

$$= \left[x^2+8x\right]_1^2 + \left[3x^2\right]_2^4$$

$$\Rightarrow\ I = \left[\left(2^2+8\cdot 2\right) - \left(1^2+8\cdot 1\right)\right] + \left[\left(3{\cdot}2^2\right) - \left(3\cdot 4^2\right)\right]$$

$$\Rightarrow\ I = [20-9] + [12-48] = 11 - 36 = -25$$

Exercise: ***Evaluate the following integration.***

EX.1. $\int\limits_0^1 \dfrac{dx}{2e^x - 1}$

Sol.: Let

$$I = \int\limits_0^1 \frac{dx}{2e^x-1}$$

Multiplying and dividing by e^{-x} to the integrand we get,

$$I = \int\limits_0^1 \frac{e^{-x}dx}{2e^{-x}e^x - e^{-x}}$$

$$= \int\limits_0^1 \frac{e^{-x}dx}{2-e^{-x}}$$

Now, here we observe that $\dfrac{d}{dx}(2-e^{-x}) = 0 - (-e^{-x}) = e^{-x}$

$$I = \int\limits_0^1 \frac{e^{-x}dx}{2-e^{-x}} = \int\limits_0^1 \frac{\frac{d}{dx}(2-e^{-x})dx}{2-e^{-x}}$$

Using the formula of integration: $\int \dfrac{f'(x)}{f(x)}\ dx = \log|f(x)|$, we get,

$$I = \left[\log(2-e^{-x})\right]_0^1$$

$$\Rightarrow I = \left[\log(2-e^{-1}) - \log(2-e^0)\right]$$

$$\Rightarrow I = \left[\log(2 - \frac{1}{e}) - \log(2-1)\right]$$

$$\Rightarrow I = \left[\log(\frac{2e-1}{e}) - \log 1\right]$$

$$\Rightarrow I = \log\left(\frac{2e-1}{e}\right)$$

EX.2. $\int\limits_0^{\frac{\pi}{2}} \sqrt{\sin\theta} \cdot \cos^5\theta \ d\theta$

Sol.: Let

$$I = \int\limits_0^{\frac{\pi}{2}} \sqrt{\sin\theta} \cdot \cos^5\theta \ d\theta$$

$$For \int \ \sin^m x \cdot \cos^n x \ dx$$

we select substitution
$t = \sin x$ when n is odd positive number,
and $t = \cos x$ when m is odd positive number.
so, we take substitution $t = \sin\theta$
$\Rightarrow dt = \cos\theta \ d\theta$

Further limits of the integral also change, when $\theta = 0 \ \Rightarrow t = \sin 0 = 0$ and when $\theta = \frac{\pi}{2} \ \Rightarrow \ t = \sin\frac{\pi}{2} = 1$ and $\cos^2\theta = 1 - \sin^2\theta \ \Rightarrow \cos^2\theta = 1 - t^2$
So, we have

$$I = \int\limits_0^{\pi/2} \sqrt{\sin\theta} \cdot (\cos^2\theta)^2 \cos\theta \ d\theta$$

with the substitution we get,

$$\Rightarrow I = \int\limits_0^1 \sqrt{t} \cdot (1-t^2)^2 \ dt$$

$$\Rightarrow I = \int\limits_0^1 t^{1/2} \cdot (1-t^2)^2 \ dt$$

$$\Rightarrow I = \int\limits_0^1 t^{1/2} \cdot (1 - 2t^2 + t^4) \ dt$$

$$\Rightarrow I = \int\limits_0^1 (t^{1/2} - 2t^{1/2}t^2 + t^{1/2}t^4) \ dt$$

$$\Rightarrow I = \int\limits_0^1 (t^{1/2} - 2t^{5/2} + t^{9/2}) \ dt$$

$$\Rightarrow I = \left[\frac{t^{3/2}}{3/2} - \frac{2t^{7/2}}{7/2} + \frac{t^{11/2}}{11/2}\right]_0^1$$

$$\Rightarrow I = \left[\frac{2t^{3/2}}{3} - \frac{4t^{7/2}}{7} + \frac{2t^{11/2}}{11}\right]_0^1$$

$$\Rightarrow I = \left[\frac{2(1)^{3/2}}{3} - \frac{4(1)^{7/2}}{7} + \frac{2(1)^{11/2}}{11}\right] - \left[\frac{2(0)^{3/2}}{3} - \frac{4(0)^{7/2}}{7} + \frac{2(0)^{11/2}}{11}\right]$$

$$\Rightarrow I = \frac{2}{3} - \frac{4}{7} + \frac{2}{11}$$

$$\Rightarrow I = \frac{154 - 132 + 42}{231} = \frac{64}{231}$$

EX.3. $\int_0^{\frac{\pi}{4}} \frac{dx}{4\sin^2 x + 5\cos^2 x}$

Sol.: Let

$$I = \int_0^{\frac{\pi}{4}} \frac{dx}{4\sin^2 x + 5\cos^2 x}$$

Dividing by $\cos^2 x$ the denominator and numerator of the integrand, we get,

$$I = \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{4\tan^2 x + 5}\ dx$$

Taking substitution $t = \tan x \quad \Rightarrow dt = \sec^2 x\ dx$

Further limits of the integral also change, when $x = 0\ \Rightarrow t = \tan 0 = 0$ and when $x = \frac{\pi}{4}\ \Rightarrow\ t = \tan\frac{\pi}{4} = 1$
So, we have

$$I = \int_0^1 \frac{dt}{4t^2 + 5}$$

$$I = \int_0^1 \frac{dt}{4(t^2 + \frac{5}{4})} = \frac{1}{4}\int_0^1 \frac{dt}{(t^2 + (\frac{\sqrt{5}}{2})^2)}$$

By the formula: $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\ \tan^{-1}\left(\frac{x}{a}\right)$, we get,

$$I = \frac{1}{4} \cdot \left[\frac{1}{\sqrt{5}/2}\ \tan^{-1}\left(\frac{t}{\sqrt{5}/2}\right)\right]_0^1$$

$$\Rightarrow I = \frac{\sqrt{5}}{2}\ \left[\tan^{-1}\left(\frac{2t}{\sqrt{5}}\right)\right]_0^1$$

$$\Rightarrow I = \frac{\sqrt{5}}{2}\ \left[\tan^{-1}\left(\frac{2\cdot 1}{\sqrt{5}}\right) - \tan^{-1}\left(\frac{2\cdot 0}{\sqrt{5}}\right)\right]$$

$$\Rightarrow I = \frac{\sqrt{5}}{2}\ \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

EX.4. $\int_0^{\frac{\pi}{4}} \sec x\ dx$

Sol.: Let

$$I = \int_0^{\frac{\pi}{4}} \sec x \; dx$$

We know the formula $\int \sec x \; dx = \log|\sec x + \tan x|$, Using that here, we get,

$$I = \int_0^{\pi/4} \sec x \; dx = [\log(\sec x + \tan x)]_0^{\pi/4}$$

$$= [\log(\sec(\pi/4) + \tan(\pi/4))] - [\log(\sec 0 + \tan 0)]$$

$$= \left[\log(\sqrt{2} + 1)\right] - [\log(1 + 0)]$$

Hence,

$$I = \left[\log(\sqrt{2} + 1)\right]$$

EX.5. $\int_3^5 \frac{x^2}{x^2 - 4} \; dx$

Sol.: Let

$$I = \int_3^5 \frac{x^2}{x^2 - 4} \; dx$$

$$I = \int_3^5 \frac{x^2 - 4 + 4}{x^2 - 4} \; dx$$

$$I = \int_3^5 \frac{x^2 - 4}{x^2 - 4} \; dx + \int_3^5 \frac{4}{x^2 - 4} \; dx$$

$$I = \int_3^5 1 \; dx + 4 \int_3^5 \frac{1}{x^2 - 2^2} \; dx$$

$$I = (x)_3^5 + 4 \cdot \left[\frac{1}{2 \cdot 2} \log\left(\frac{x - 2}{x + 2}\right)\right]_3^5$$

$$I = (5 - 3) + \{\log\left(\frac{5 - 2}{5 + 2}\right) - \log\left(\frac{3 - 2}{3 + 2}\right)\}$$

$$I = 2 + \{\log\left(\frac{3}{7}\right) - \log\left(\frac{1}{5}\right)\}$$

$$I = 2 + \log\left(\frac{15}{7}\right)$$

EX.6. $\int_0^1 \frac{dx}{\sqrt{x^2 + 4x + 3}}$

Sol.: Let

$$I=\int_0^1 \frac{dx}{\sqrt{x^2+4x+3}}$$

$$I=\int_0^1 \frac{dx}{\sqrt{x^2+4x+4-1}}$$

$$I=\int_0^1 \frac{dx}{\sqrt{(x+2)^2-1}}$$

Taking substitution $y=x+2 \quad \Rightarrow dy=dx$ and when $x=0 \ \Rightarrow y=2$ and when $x=1 \ \Rightarrow y=3$.

$$I=\int_2^3 \frac{dy}{\sqrt{y^2-1}}$$

Using the formula of integration: $\int \frac{dx}{\sqrt{x^2-a^2}}=\log(x+\sqrt{x^2-a^2})$, we get,

$$I=\left[\log(y+\sqrt{y^2-1})\right]_2^3$$

$$I=\left[\log(3+\sqrt{3^2-1})\right]-\left[\log(2+\sqrt{2^2-1})\right]$$

$$I=\left[\log(3+\sqrt{8})\right]-\left[\log(2+\sqrt{3})\right]$$

$$I=\log\left[\frac{(3+2\sqrt{2})}{(2+\sqrt{3})}\right]$$

⋙3.5. *Integration By Parts method in case of Definite Integration.*

We have studied the the integration by parts method in the previous unit which is stated as follows:

$$\int u\ v\ dx=u\left(\int\ v\ dx\right)-\int \frac{d}{dx}(u)\cdot\left(\int\ v\ dx\right)\ dx$$

But in the case of the definite integral, we have

$$\int_a^b u\ v\ dx=\left[u\left(\int\ v\ dx\right)\right]_a^b-\int_a^b \frac{d}{dx}(u)\cdot\left(\int\ v\ dx\right)\ dx$$

As for example,

Let us consider the integral

$$I=\int_0^1 x\cdot e^x\ dx$$

Here choosing $u=x$ and $v=e^x$ Following the LIATE rule For the choice of the function "u" we follow the priority order : **LIATE** *(L: Logarithmic function, I: Inverse function, A: Algebraic function, T: Trigonometric function, E: Exponential function.)*

$Log(\log x, \log(x+1), ...)$
$\gg Inverse(\tan^{-1}, \cos^{-1}, \sin^{-1}, ..)$
$\gg Algebraic(1, x, x^2, 1+x, 2+7x, x^3, ...)$
$\gg Trigonometric(\sin, \cos, \tan, ...)$
$\gg Exponential(e^x, e^{x+1}, ...)$

and applying the above rule, we get,

$$I = \left[x\left(\int e^x\ dx\right)\right]_0^1 - \int\limits_0^1 \frac{d}{dx}(x) \cdot \left(\int e^x\ dx\right)\ dx$$

$$I = [x \cdot e^x]_0^1 - \int\limits_0^1 (1) \cdot e^x\ dx$$

$$I = \left[1 \cdot e^1 - 0 \cdot e^0\right] - [\ e^x\ ]_0^1$$

$$I = e - [e^1 - e^0] = e - (e-1) = 1$$

Exercise: ***Evaluate the following integration.***

Ex.1. $\int\limits_0^1 \tan^{-1} x\ dx$

Sol.: Let

$$I = \int\limits_0^1 \tan^{-1} x\ dx$$

Using the integration by parts method taking $u = \tan^{-1} x$ and $v = 1$, we get,

$$I = \left[\tan^{-1} x \int 1\ dx\right]_0^1 - \int\limits_0^1 \frac{d}{dx}(\tan^{-1} x) \cdot (\int 1\ dx)\ dx$$

$$I = \left[x \tan^{-1} x\right]_0^1 - \int\limits_0^1 \frac{1}{1+x^2} \cdot xdx$$

$$I = \left[1 \cdot \tan^{-1} 1 - 0 \cdot \tan^{-1} 0\right] - \int\limits_0^1 \frac{x}{1+x^2}dx$$

$$I = \left[1 \cdot \tan^{-1} 1 - 0\right] - \frac{1}{2}\int\limits_0^1 \frac{2x}{1+x^2}dx$$

$$I = \tan^{-1} 1 - \frac{1}{2}\int\limits_0^1 \frac{2x}{1+x^2}dx$$

$$I = \frac{\pi}{4} - \frac{1}{2}\int\limits_0^1 \frac{\frac{d}{dx}(1+x^2)}{1+x^2}dx$$

$$I = \frac{\pi}{4} - \frac{1}{2}\left[\log(1+x^2)\right]_0^1$$

$$I = \frac{\pi}{4} - \frac{1}{2}\left[\log(1+1^2) - \log(1+0^2)\right]$$

$$I = \frac{\pi}{4} - \frac{1}{2}\log 2$$

Ex.2. $\int\limits_0^{1/2} \frac{sin^{-1}x}{(1-x^2)^{3/2}}\ dx$

Sol.: Let

$$I = \int\limits_0^{1/2} \frac{sin^{-1}x}{(1-x^2)^{3/2}}\ dx$$

Use the substitution $\theta = \sin^{-1} x \quad \Rightarrow \ d\theta = \dfrac{1}{\sqrt{1-x^2}}\ dx$

Also, when $x = 0 \Rightarrow \theta = \sin^{-1} 0 = 0$ and when $x = \frac{1}{2} \ \Rightarrow \theta = \sin^{-1}(1/2) = \frac{\pi}{6}$

So, with this we have,

$$I = \int\limits_0^{1/2} \frac{sin^{-1}x\ dx}{(1-x^2)\sqrt{(1-x^2)}}$$

$$\Rightarrow I = \int\limits_0^{\pi/6} \frac{\theta}{\cos^2\theta}\ d\theta$$

$$\Rightarrow I = \int\limits_0^{\pi/6} \theta \cdot \sec^2\theta\ d\theta$$

Now, we apply the Integration by parts method with $u = \theta$ and $v = \sec^2\theta$, we get,

$$\Rightarrow I = \left[\theta \cdot \left(\int \sec^2\theta\ d\theta\right)\right]_0^{\pi/6} - \int\limits_0^{\pi/6} \frac{d}{d\theta}(\theta) \cdot \left(\int \sec^2\theta\ d\theta\right)\ d\theta$$

$$\Rightarrow I = [\theta \cdot (\tan\theta)]_0^{\pi/6} - \int\limits_0^{\pi/6} (1) \cdot (\tan\theta)\ \ d\theta$$

$$\Rightarrow I = \left[\frac{\pi}{6} \cdot \left(\tan\frac{\pi}{6}\right) - 0 \cdot (\tan 0)\right] - \int\limits_0^{\pi/6} \tan\theta\ d\theta$$

$$\Rightarrow I = \frac{\pi}{6} \cdot \frac{1}{\sqrt{3}} - [\ \log(\sec x)\ ]\ _0^{\pi/6}$$

$$\Rightarrow I = \frac{\pi}{6\sqrt{3}} - [\ \log(\sec\pi/6) - \log(\sec 0)\ ]$$

$$\Rightarrow I = \frac{\pi}{6\sqrt{3}} - [\ \log(\frac{2}{\sqrt{3}}) - \log(1)\ ]$$

$$\Rightarrow I = \frac{\pi}{6\sqrt{3}} - \log(\frac{2}{\sqrt{3}})$$

Ex.3. $\int\limits_0^{\sqrt{2}} x^3 e^{x^2}\ dx$

Sol.: Let

$$I = \int\limits_0^{\sqrt{2}} x^3 e^{x^2}\ dx$$

Here taking the substitution $t = x^2\ \Rightarrow dt = 2x\ dx$ and when $x = 0 \Rightarrow t = 0,\ x = \sqrt{2}\ \Rightarrow t = 2$

With this and

$$I = \int\limits_0^{\sqrt{2}} x^2 e^{x^2}\ x\ dx$$

We get,

$$I = \int\limits_0^{2} t e^t\ dt$$

Next, we apply the Integration by parts method with $u = t$ and $v = e^t$

$$I = \left[t \cdot e^t\right]_0^2 - \int\limits_0^2 (1) \cdot e^t\ dt$$

$$I = \left[2 \cdot e^2 - 0 \cdot e^0\right] - [\ e^t\ ]_0^2$$

$$I = [2e^2] - [\ e^2 - e^0\ ]$$

$$I = 2e^2 -\ e^2 + 1 = e^2 + 1$$

Ex.4. $\int\limits_0^{\frac{\pi}{2}} x^2 \cos 2x\ dx$

Sol.: Let

$$I = \int\limits_0^{\frac{\pi}{2}} x^2 \cos 2x\ dx$$

Using the integration by parts method for $u = x^2$ and $v = \cos 2x$, we get

$$I = \left[x^2 \left(\int \cos 2x\ dx\right)\right]_0^{\pi/2} - \int\limits_0^{\pi/2} \frac{d}{dx}(x^2) \left(\int \cos 2x\ dx\right)\ dx$$

$$I = \left[x^2 \cdot \frac{\sin 2x}{2}\right]_0^{\pi/2} - \int\limits_0^{\pi/2} (2x) \cdot \frac{\sin 2x}{2}\ dx$$

$$I = \left[(\pi/2)^2 \cdot \frac{\sin 2(\pi/2)}{2} - (0)^2 \cdot \frac{\sin 2(0)}{2}\right] - \int_0^{\pi/2} x \cdot \sin 2x \; dx$$

$$I = \left[(\pi/2)^2 \cdot \frac{\sin(\pi)}{2} - 0\right] - \int_0^{\pi/2} x \cdot \sin 2x \; dx$$

$$I = 0 - \int_0^{\pi/2} x \cdot \sin 2x \; dx = -\int_0^{\pi/2} x \cdot \sin 2x \; dx$$

Once again applying the integration by parts method for $u = x$ and $v = \sin 2x$, we get,

$$I = -\left[x\left(\int \sin 2x \; dx\right)\right]_0^{\pi/2} + \int_0^{\pi/2} (1)\left(\int \sin 2x \; dx\right) \; dx$$

$$I = -\left[x\frac{-\cos 2x}{2}\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{-\cos 2x}{2} \; dx$$

$$I = \left[\pi/2 \cdot \frac{-\cos 2\pi/2}{2} - 0 \cdot \frac{-\cos 2 \cdot 0}{2}\right] - \frac{1}{2}\int_0^{\pi/2} \cos 2x \; dx$$

$$I = -\left[\pi/2 \cdot \frac{-\cos \pi}{2}\right] - \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$I = -\left[\frac{\pi}{2} \cdot \frac{-(-1)}{2}\right] - \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$I = -\left[\frac{\pi}{2} \cdot \frac{-(-1)}{2}\right] - \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$I = -\frac{\pi}{4} - \frac{1}{2}\left[\frac{\sin 2(\pi/2)}{2} - \frac{\sin 2(0)}{2}\right]$$

$$I = -\frac{\pi}{4} - \frac{1}{2}\left[\frac{\sin(\pi)}{2} - \frac{\sin(0)}{2}\right] = -\frac{\pi}{4}$$

Ex.5. $\int_{-1}^{1} \sin^3 x \cos^4 x \; dx$

Sol.: Let

$$I = \int_{-1}^{1} \sin^3 x \cos^4 x \; dx$$

Here, we observe that the integrand function $f(x) = \sin^3 x \cos^4 x$ is such that $f(-x) = (\sin(-x))^3(\cos(-x))^4 = (-\sin x)^3(\cos x)^4 = -\sin^3 x \cos^4 x = -f(x) \Rightarrow f(-x) = -f(x)$

Hence, given integrand is an odd function.

So, by the result for an odd function: $\int\limits_{-a}^{a} f(x)\, dx = 0$, we have

$$I = 0$$

Ex.6. $\int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 x\, dx$

Sol.: Let

$$I = \int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 x\, dx$$

Here, we observe that cos is an even function so by the result for the even function: $\int\limits_{-a}^{a} f(x)dx = 2\int\limits_{0}^{a} f(x)dx$, we get,

$$I = 2\int\limits_{0}^{\frac{\pi}{4}} \cos^2 x\, dx$$

$$I = 2\int\limits_{0}^{\frac{\pi}{4}} \left(\frac{1+\cos 2x}{2} \right) dx$$

$$I = \frac{2}{2}\int\limits_{0}^{\frac{\pi}{4}} (1+\cos 2x) \, dx$$

$$I = \left(x + \frac{\sin 2x}{2} \right)_{0}^{\frac{\pi}{4}}$$

$$I = \left(\frac{\pi}{4} + \frac{\sin 2(\frac{\pi}{4})}{2} \right) - \left(0 + \frac{\sin 2(0)}{2} \right)$$

$$I = \left(\frac{\pi}{4} + \frac{1}{2} \right) = \frac{\pi+2}{4}$$

$\preccurlyeq \overbrace{\underbrace{EX}}\succcurlyeq$. If $\int\limits_{0}^{k} \frac{dx}{2+8x^2} = \frac{\pi}{16}$, then find k.

Sol.: Here

$$\int\limits_{0}^{k} \frac{dx}{2+8x^2} = \frac{\pi}{16}$$

$$\Rightarrow \frac{1}{8}\int\limits_{0}^{k} \frac{dx}{\frac{1}{4}+x^2} = \frac{\pi}{16}$$

$$\Rightarrow \int\limits_{0}^{k} \frac{dx}{x^2+\frac{1}{4}} = \frac{\pi}{2}$$

$$\Rightarrow \int\limits_{0}^{k} \frac{dx}{x^2+\frac{1}{4}} = \frac{\pi}{2}$$

Using the formula $\int \dfrac{dx}{x^2+a^2} = \dfrac{1}{a}\tan^{-1}\frac{x}{a}$, we get

$$\Rightarrow \left[\frac{1}{1/2}\tan^{-1}\frac{x}{1/2}\right]_0^k = \frac{\pi}{2}$$

$$\Rightarrow 2\left[\tan^{-1}2x\right]_0^k = \frac{\pi}{2} \quad \Rightarrow \quad \left[\tan^{-1}2x\right]_0^k = \frac{\pi}{4}$$

$$\Rightarrow \left[\tan^{-1}2k - \tan^{-1}(2\cdot 0)\right] = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}2k = \frac{\pi}{2} \qquad \Rightarrow \quad 2k = \tan\frac{\pi}{4}$$

$$\Rightarrow 2k = 1 \quad \Rightarrow k = \frac{1}{2}$$

≺ EX ≻ . Prove that $\int\limits_0^{2a} \dfrac{f(x)}{f(x)+f(2a-x)}\, dx = a$

Sol.: Let

$$I = \int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)}\, dx$$

We know the result for the definite integral: $\int\limits_a^b f(x)\, dx = \int\limits_a^b f(a+b-x)\, dx$, we get

$$\int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)}\, dx = \int_0^{2a} \frac{f(2a-x)}{f(2a-x)+f(2a-(2a-x))}\, dx$$

Therefore,

$$\int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)}\, dx = \int_0^{2a} \frac{f(2a-x)}{f(2a-x)+f(x)}\, dx$$

So, we have

$$\int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)}\, dx = I = \int_0^{2a} \frac{f(2a-x)}{f(2a-x)+f(x)}\, dx$$

$$\int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)}\, dx + \int_0^{2a} \frac{f(2a-x)}{f(2a-x)+f(x)}\, dx = 2I$$

$$\int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)} + \frac{f(2a-x)}{f(2a-x)+f(x)}\, dx = 2I$$

$$\Rightarrow \int_0^{2a} \frac{f(x)+f(2a-x)}{f(x)+f(2a-x)}\, dx = 2I$$

$$\Rightarrow \int_0^{2a} dx = 2I$$

$$\Rightarrow [\ x\ ]_0^{2a} = 2I$$

$$\Rightarrow [2a-0] = 2I \Rightarrow 2a = 2I \Rightarrow I = a$$

Therefore,

$$\int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)}\ dx = a$$

$\preccurlyeq \underbrace{\overbrace{EX}}\succcurlyeq$. If $f(x) = f(a+b-x)$, Prove that $\int_a^b xf(x)\ dx = \frac{(a+b)}{2}\int_a^b f(x)\ dx$

Sol.: We know the result for definite integral as:

$$\int_a^b g(x)\ dx = \int_a^b g(a+b-x)\ dx$$

Here, taking $g(x) = x \cdot f(x)\ \Rightarrow g(a+b-x) = (a+b-x)\cdot f(a+b-x)$

But it is given that $f(x) = f(a+b-x)$

This implies, $g(a+b-x) = (a+b-x)\cdot f(x)$

Hence,

$$\int_a^b x\cdot f(x)\ dx = \int_a^b (a+b-x)\cdot f(x)\ dx$$

$$\Rightarrow \int_a^b x\cdot f(x)\ dx = \int_a^b ((a+b)-x)\cdot f(x)\ dx$$

$$\Rightarrow \int_a^b x\cdot f(x)\ dx = \int_a^b (a+b)\cdot f(x)\ dx - \int_a^b x\cdot f(x)\ dx$$

$$\Rightarrow \int_a^b x\cdot f(x)\ dx + \int_a^b x\cdot f(x)\ dx = \int_a^b (a+b)\cdot f(x)\ dx$$

$$\Rightarrow 2\int_a^b x\cdot f(x)\ dx = (a+b)\int_a^b f(x)\ dx$$

$$\Rightarrow \int_a^b x\cdot f(x)\ dx = \frac{(a+b)}{2}\int_a^b f(x)\ dx$$

EXTRA EXAMPLES :

(1) Evaluate the following integration.

(a) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cot x\ dx$

(b) $\int_0^2 \frac{6x+3}{x^2+4}\ dx$

(c) $\int\limits_{\frac{\pi}{2}}^{\pi} \dfrac{1-\sin x}{1-\cos x}\ dx$

(d) $\int\limits_{-1}^{1} \dfrac{x^3}{a^2-x^2}\ dx \quad a>1$

(e) $\int\limits_{0}^{\pi} \sin^4 x \cos^3 x\ dx$

(f) $\int\limits_{0}^{2\pi} \sin^3 x \cos^2 x\ dx$

(2) Evaluate $\int\limits_{-1}^{1} f(x)\ dx$,where $f(x) = \begin{cases} 1-2x, & -1 \le x \le 0 \\ 1+2x, & 0 \le x \le 1 \end{cases}$

(3) If $\int\limits_{0}^{k} \dfrac{\tan x}{1+\tan x}\ dx = \dfrac{\pi}{4}$, then find k.

UNIT - 4: US02EMTH02 (ELECTIVE MATHS, SEM. II)

Differential Equations

Reference book : Gujarat State Board of School Textbooks , Standard - 12 MATHEMATICS - 2 ,(CHAPTER - 8)

"Have you ever thought 'why should students of Biology study mathematics?'. In the present time mathematics is being widely used. By use of mathematics, representation of any subject becomes clear and well-defined and still the expression becomes compact. 'Differential Equations' is also a branch of mathematics. Most of the branches of science and management make use of differential equations. In fact, this branch provides one of the most powerful tools in the mathematics.In the present chapter we shall procure an elementary information about differential equations."

4.1 *Differential Equation: Definition*

The equation involving derivative/s of dependent variable with respect to the independent variable and together with the variables is called differential equation.

More precisely if $y = f(x)$ is the dependent variable depending on independent variable x, then the equation involving $x, y, \dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, ...$ is called differential equation.

i.e., a function $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ...\right) = 0.$

⁂ *Practical Example of Differential Equation:-*

Let us consider the problem from your own world that is Life science.
The rate of growth of bacteria is proportional to their number present at a moment(which is practically established by the available experimental data).

Thus if at time t the number of bacteria is x then this problem can be described by a differential equation

$$\frac{dx}{dt} = kx$$

where k is a constant of proportionality.

This is an equation, to determine number of bacteria at any moment.

4.2 *Order and Degree of Differential Equation*

If a differential equation is written in the form of a polynomial the order of the highest order derivative occurring in the equation is called order of the differential equation and its power is called the degree of the differential equation.

Exercise: **Obtain the order and degree of the following differential equation.**

1. $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 + xy = 0$

Sol.:- The highest order derivative in the above equation is $\dfrac{d^2y}{dx^2}$ and its degree(power) is 1.
$\therefore$ The differential equation has order 2 and degree 1.

2. $\dfrac{d^2y}{dx^2} = \left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$

Sol.:- To express the above differential equation in to the polynomial form in derivatives, taking square we get,

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$$

The highest order derivative in the above equation is $\dfrac{d^2y}{dx^2}$ and its degree(power) is 2.

$\therefore$ The differential equation has order 2 and degree 2.

3. $\sqrt{1-y^2}dx + \sqrt{1-x^2}dy = 0$

Sol.:- Expressing the above equation in the polynomial form in terms of derivatives, we get,

$$\sqrt{1-y^2} + \sqrt{1-x^2}\frac{dy}{dx} = 0$$

The highest order derivative in the above equation is $\dfrac{dy}{dx}$ and its degree(power) is 1.
$\therefore$ The differential equation has order 1 and degree 1.

4. $\sin\left(\dfrac{dy}{dx}\right) + 5y = 9$

Sol.:- Expressing the above equation in the polynomial form in terms of derivatives, we get,

$$\sin\left(\frac{dy}{dx}\right) = 9 - 5y \quad \Rightarrow \left(\frac{dy}{dx}\right) = \sin^{-1}(9-5y)$$

The highest order derivative in the above equation is $\dfrac{dy}{dx}$ and its degree(power) is 1.
$\therefore$ The differential equation has order 1 and degree 1.

5. $\dfrac{d^2y}{dx^2} + 3y = 0$

Sol.:- The highest order derivative in the above equation is $\dfrac{d^2y}{dx^2}$ and its degree(power) is 1.
$\therefore$ The differential equation has order 2 and degree 1.

6. $\dfrac{dy}{dx} + \dfrac{1}{(\frac{dy}{dx})} = 5$

Sol.:- To express the above differential equation in to the polynomial form in derivatives, taking L.C.M= $\dfrac{dy}{dx}$ we get,

$\Rightarrow \left(\dfrac{dy}{dx}\right)^2 + 1 = 5\dfrac{dy}{dx}$

The highest order derivative in the above equation is $\dfrac{dy}{dx}$ and its degree(power) is 2.

$\therefore$ The differential equation has order 1 and degree 2.

7. $x + \dfrac{dy}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

Sol.:- To express the above differential equation in to the polynomial form in derivatives, taking square we get,

$\left(x + \dfrac{dy}{dx}\right)^2 = 1 + \left(\dfrac{dy}{dx}\right)^2$

The highest order derivative in the above equation is $\dfrac{dy}{dx}$ and its degree(power) is 2.

$\therefore$ The differential equation has order 1 and degree 2.

$\gg\!\rightarrow$ *Origin of Differential Equation:*

Each family of curves have its differential equation, which is obtained by eliminating the arbitrary constants from the given equation of the family.

We follow the simple rule for obtaining the differential equation of the given family.

If the equation of the family contains one arbitrary constant then we have to differentiate it once, If it involves two arbitrary constants then we have to differentiate the equation twice, and so on.

Some times it is easier to eliminate the arbitrary constants by differentiation, but when it is not eliminated by direct differentiation then find the values of those constants in terms of the derivative and variables then substitute into the equation of the family.

Ex. Obtain the differential equation of family of all the parallel lines represented by y = 2x + c with slop 2 .(c is arbitrary constant)

Sol.:- Here $y = 2x + c$, differentiating with respect to x, we get

$$\frac{d}{dx}(y) = 2\frac{d}{dx}(x) + \frac{d}{dx}(c)$$

$$\Rightarrow \frac{dy}{dx} = 2$$

which is the required differential equation.

Ex. Obtain the differential equation representing all lines of family y = mx + c .(m and c are arbitrary constants)

Sol.:- Here $y = mx + c$, differentiating with respect to x, we get

$$\frac{d}{dx}(y) = m\frac{d}{dx}(x) + \frac{d}{dx}(c)$$

$$\Rightarrow \frac{dy}{dx} = m$$

Further differentiating we get

$$\Rightarrow \frac{d^2y}{dx^2} = 0$$

which is the required differential equation.

Ex. Obtain the differential equation of family of circles having centre on x - axis and radius 1 unit.

Sol.:- The general equation of the circle with center (a, b) and radius r is

$$(x - a)^2 + (y - b)^2 = r^2$$

The point on x-axis have its y-coordinate=0.
As it is given here the center of the circle lies on the x-axis so its y - coordinate =0, i.e., (a,0) be the coordinates of center.

And r denotes the radius in the general equation, here it is given that the radius is 1, $r = 1$.
So, the equation of the circle with center on x-axis and radius 1 is given by,

$$(x - a)^2 + y^2 = 1$$

Now, to get its differential equation we are going to differentiate it with respect to x,

By differentiating with respect to x, we get,

$$2(x - a)\frac{d}{dx}(x - a) + 2y\frac{d}{dx}(y) = 0$$

$$\Rightarrow (x - a)(1 - 0) + y\frac{dy}{dx} = 0$$

$$\Rightarrow (x - a) + y\frac{dy}{dx} = 0$$

$$\Rightarrow a = x + y\frac{dy}{dx}$$

Using this value of a into the equation $(x - a)^2 + y^2 = 1$, we get,

$$\left(x - (x + y\frac{dy}{dx})\right)^2 + y^2 = 1$$

$$\Rightarrow \left(-y\frac{dy}{dx}\right)^2 + y^2 = 1$$

$$\Rightarrow y^2\left[\left(\frac{dy}{dx}\right)^2 + 1\right] = 1$$

which is the required differential equation.

Ex. Obtain the differential equation of family of curves $y = a\sin(x+b)$, a and b are arbitrary constants .

Sol.:- Here, $y = a\sin(x+b)$ is the given equation.

Differentiating with respect to x, we get,

$$\frac{dy}{dx} = a\cos(x+b)\ \frac{d}{dx}(x+b)$$

$$\Rightarrow \frac{dy}{dx} = a\cos(x+b)\cdot(1+o)$$

$$\Rightarrow \frac{dy}{dx} = a\cos(x+b)$$

Further differentiating with respect to x, we get,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = -a\sin(x+b)\frac{d}{dx}(x+b)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -a\sin(x+b)$$

But we have $a\sin(x+b) = y$ as it is given.

So,

$$\frac{d^2y}{dx^2} = -y \Rightarrow \frac{d^2y}{dx^2} + y = 0$$

which is the required differential equation.

4.3 *Solution of the Differential Equation*

Let a differential equation in variables x and y be given. If we can find a function $y = f(x)$ such that x, y and its derivatives identically satisfy the differential equation, the function $y = f(x)$ is called a solution of differential equation.

We know that if $f'(x) = F(x)$ then $[f(x)+c]' = F(x)$, where c is an arbitrary constant. Thus we will get a family of solutions. The solution of a differential equation covering all its solutions is called **the general solution** *of the differential equation. If we can obtain definite value of c because of given values of x, y and the derivatives, we obtain a* **particular solution** *of the differential equation.*

The conditions are called initial conditions.

Note: The general solution of a differential equation will contain as many constants as the order of the differential equation.

Ex. Verify that $y = e^x,\ x \in \mathbb{R}$ is a solution of the differential equation $\frac{dy}{dx} = y$.

Sol.:- Here, we have $y = e^x$. To show that it is a solution of the differential equation

$$\frac{dy}{dx} = y$$

for that we have to show that if we put $y = e^x$ into the differential equation $\frac{dy}{dx} = y$ it should be verified/satisfied.

Let,

$$y = e^x$$

Differentiating it with respect to x, we get,

$$\frac{d}{dx}(y) = \frac{d}{dx}(e^x) \Rightarrow \frac{dy}{dx} = e^x$$

and

$$y = e^x$$

$$\Rightarrow \frac{dy}{dx} = y$$

Hence, $y = e^x$ is a solution of differential equation $\frac{dy}{dx} = y$.

Ex. Verify that $y = \sin x,\ x \in \mathbb{R}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Sol.:- Here, $y = \sin x$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \cos x$$

Further, differentiating the above equation with respect to x, we get

$$\frac{d^2y}{dx^2} = -\sin x$$

Also, it is provided that $y = \sin x$

$$L.H.S. = \frac{d^2y}{dx^2} + y = -\sin x + \sin x = 0 = R.H.S.$$

So, it is verified that $y = \sin x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$

Ex. Verify that $y = \cos x,\ x \in \mathbb{R}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Sol.:- Here, $y = \cos x$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -\sin x$$

Further, differentiating the above equation with respect to x, we get

$$\frac{d^2y}{dx^2} = -\cos x$$

Also, it is provided that $y = \cos x$

$$L.H.S. = \frac{d^2y}{dx^2} + y = -\cos x + \cos x = 0 = R.H.S.$$

So, it is verified that $y = \cos x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$

Ex. Verify that $y = ax + a^2$,(a is arbitrary constant) is the general solution of the differential equation $\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) = y$.

Sol.:- Here, we have $y = ax + a^2$

Differentiating with respect to x-axis, we get

$$\frac{dy}{dx} = a + 0 \ \Rightarrow a = \frac{dy}{dx}$$

L.H.S.$=\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) = (a)^2 + x\,(a)$

$$= ax + a^2 = y = R.H.S.$$

So, we have verified that $y = ax + a^2$ is the general solution of the differential equation $\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) = y$.

Ex. Verify that $y = cx + \frac{1}{c}$ is the general solution of the differential equation $y\left(\frac{dy}{dx}\right) = x\left(\frac{dy}{dx}\right)^2 + 1$,where c is arbitrary constant.

Sol.:- Here $y = cx + \frac{1}{c}$, differentiating with respect to x, we get

$$\frac{d}{dx}(y) = c\frac{d}{dx}(x) + \frac{d}{dx}(\frac{1}{c})$$
$$\Rightarrow \frac{dy}{dx} = c$$

Here we have to verify that

$$y\left(\frac{dy}{dx}\right) = x\left(\frac{dy}{dx}\right)^2 + 1$$

L.H.S.$=y\left(\frac{dy}{dx}\right) = (cx + \frac{1}{c}) \cdot c = c^2x + 1$

R.H.S.$=x\left(\frac{dy}{dx}\right)^2 + 1 = x \cdot (c)^2 + 1 = c^2x + 1$

$\Rightarrow$ L.H.S.=R.H.S.

Hence, it is verified that $y = cx + \frac{1}{c}$ is the general solution of the differential equation $y\left(\frac{dy}{dx}\right) = x\left(\frac{dy}{dx}\right)^2 + 1$

4.4 *Differential Equation of* 1^{st} *order and* 1^{st} *degree*

The general form of differential equation of first order and first degree is

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \Rightarrow \; M(x,y)dx + N(x,y)dy = 0.$$

$$i.e., Mdx + Ndy = 0.$$

Now we will discuss one of the methods to solve such equations.

4.5 *Variable Separable method*

This is a method of separating variables x and y. We know first order and first degree equation is

$$Mdx + Ndy = 0.$$

If $M(x,y)$ is a function of x alone and $N(x,y)$ is a function of y alone the equation is said to be in variable separable form.

We can take $M = f(x)$ and $N = f(y)$

That gives the form of the equation as

$$f(x)dx + g(y)dy = 0$$

Whose solution is obtained by taking integration as follows

$$\int f(x)dx + \int g(y)dy = c$$

where c is a constant of integration.

This is known as the general solution of the given differential equation.

$\ll$ **Note** : $\gg$

(i) c can be any arbitrary constant. According to our convenience to express solution in a suitable form we will take c as $\log c, \tan^{-1} c, e^{-c}$, ... etc.

(ii) Variables are separable means all terms containing x form multiplier of dx and all terms containing y form multiplier of dy.

(iii) Particular value of a constant c gives a particular solution based on some initial conditions.

Ex. Solve the differential equation $\mathbf{x(1+y^2)dx - y(1+x^2)dy = 0}$

Sol.: Here, by observing the given equation, we see that the separation method will work.

$$x(1+y^2)dx - y(1+x^2)dy = 0 \;\Rightarrow\; \frac{x}{(1+x^2)}dx - \frac{y}{(1+y^2)}dy = 0$$

Hence, variables are separable.

Now, applying integration on both sides, we get

$$\int \frac{x}{(1+x^2)}dx - \int \frac{y}{(1+y^2)}dy = c_1$$
$$\Rightarrow \frac{1}{2}\int \frac{2x}{(1+x^2)}dx - \frac{1}{2}\int \frac{2y}{(1+y^2)}dy = c_1$$
$$\Rightarrow \int \frac{2x}{(1+x^2)}dx - \int \frac{2y}{(1+y^2)}dy = 2c_1$$
$$\Rightarrow \int \frac{\frac{d}{dx}(x^2+1)}{(1+x^2)}dx - \int \frac{\frac{d}{dy}(y^2+1)}{(1+y^2)}dy = 2c_1 = c_2$$
$$\Rightarrow \log(1+x^2) - \log(1+y^2) = c_2 = \log c$$

$$\Rightarrow \log\frac{1+x^2}{1+y^2} = \log c$$

$$\Rightarrow \frac{1+x^2}{1+y^2} = c$$

Ex. Solve the differential equation $\mathbf{\frac{dy}{dx} = e^{x+y}}$.
Find the particular solution subject to initial condition , y(1)=1 .Also find y(-1).

Sol.: Here, we can rewrite the equation as

$$\frac{dy}{dx} = e^x \cdot e^y \;\Rightarrow \frac{dy}{e^y} = e^x\ dx \;\Rightarrow e^{-y}dy = e^x\ dx$$

i.e., variables are separable.

Applying integration on both sides, we get

$$\int e^{-y}dy = \int e^x\ dx + c_1$$

$$-e^{-y} = e^x + c_1 \;\Rightarrow -e^{-y} - e^x = c_1$$

$$\Rightarrow e^{-y} + e^x = -c_1 = c$$

$$\Rightarrow e^{-y} + e^x = c$$

which is the general solution of the given differential equation.

Now, it is given that $y(1) = 1$, i.e., when $x = 1$ we have $y = 1$.

Using this initial condition into the general solution $e^{-y} + e^x = c$, we get

$$e^{-1} + e^1 = c \quad \Rightarrow \frac{1}{e} + e = c \quad \Rightarrow \frac{1+e^2}{e} = c$$

So the particular solution is

$$e^{-y} + e^x = \frac{1+e^2}{e}$$

Next, we wish to determine $y(-1)$, for that put $x=-1$ into the above particular solution and find the corresponding value of y.

$$\Rightarrow e^{-y}+e^{-1}=\frac{1+e^2}{e}$$
$$\Rightarrow e^{-y}=\frac{1+e^2}{e}-e^{-1}$$
$$\Rightarrow e^{-y}=\frac{1+e^2}{e}-\frac{1}{e}$$
$$\Rightarrow e^{-y}=\frac{e^2}{e}=e=e^1 \quad \Rightarrow e^{-y}=e^1 \ \Rightarrow y=-1 \ when \ x=-1.$$

So,

$$y(-1)=-1.$$

Ex. Solve $\sec^2 x\tan y dx+\sec^2 y\tan x dy=0$. $If \ y(\pi/4)=\pi/4$, then find the particular solution of the given differential equation .

Sol.:- Here first we have to find the general solution of $\sec^2 x\tan y dx+\sec^2 y\tan x dy=0$

We can separate variables by dividing by " $\tan x\cdot\tan y$" as follows

$$\frac{\sec^2 x}{\tan x}dx+\frac{\sec^2 y}{\tan y}dy=0$$

Applying the integration, we get

$$\int\frac{\sec^2 x}{\tan x}dx+\int\frac{\sec^2 y}{\tan y}dy=\int 0$$
$$\int\frac{\frac{d}{dx}(\tan x)}{\tan x}dx+\int\frac{\frac{d}{dy}(\tan y)}{\tan y}dy=c_1$$

Using the formula $\int\frac{f'(x)}{f(x)}dx=\log(f(x))$ and adjusting $c_1=\log c$,we get

$$\log(\tan x)+\log(\tan y)=\log c$$

$$\log(\tan x\cdot\tan y)=\log c$$

$$\tan x\cdot\tan y=c$$

which is the required general solution.

Now, we determine the particular solution using the given condition $y(\pi/4)=\pi/4$

Using the above condition into the general solution,
i.e., put $x=\pi/4 \ \ and \ y=\pi/4$, we get

$$\tan(\pi/4)\cdot\tan(\pi/4)=c \quad \Rightarrow 1\cdot 1=c \quad \Rightarrow c=1$$

$$\Rightarrow \tan x\cdot\tan y=1$$

is the required particular solution.

4.6 *The equations which can be transformed into variable separable from:-*

To do such transformations we have to use the substitution $v = x + y$

EX-1: Solve $\dfrac{dy}{dx} = (x+y)^2$.

Sol.:- To transform the given equation to the variable separable form we take the substitution $v = x + y$

Differentiating with respect to x, we get

$\dfrac{dv}{dx} = 1 + \dfrac{dy}{dx} \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$ using this into the given differential equation it becomes

$$\frac{dv}{dx} - 1 = v^2$$

$$\Rightarrow \frac{dv}{dx} = 1 + v^2$$

$$\Rightarrow \frac{dv}{dx} = 1 + v^2$$

Now we can see that the variables are separable as follows

$$\frac{dv}{1+v^2} = dx$$

Applying the integration we get

$$\Rightarrow \int \frac{dv}{1+v^2} = \int dx + c$$

$$\Rightarrow \tan^{-1} v = x + c \quad \Rightarrow v = \tan(x+c)$$

Using the value of v back, we get

$$\Rightarrow x + y = \tan(x+c)$$

is the required solution.

(1) Solve $\dfrac{dy}{dx} = \sin(x+y)$

Sol.:- To transform the given equation to the variable separable form we take the substitution $v = x + y$

Differentiating with respect to x, we get

$\dfrac{dv}{dx} = 1 + \dfrac{dy}{dx} \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$ using this into the given differential equation it becomes

$$\frac{dv}{dx} - 1 = \sin v$$

$$\Rightarrow \frac{dv}{dx} = 1 + \sin v$$

$$\Rightarrow \frac{dv}{dx} = 1 + \sin v$$

Now we can see that the variables are separable as follows

$$\frac{dv}{1+\sin v} = dx$$

Applying the integration we get

$$\Rightarrow \int \frac{dv}{1+\sin v} = \int dx + c$$

$$\Rightarrow \int \frac{1}{1+\sin v} \cdot \frac{1-\sin v}{1-\sin v}\, dv = x + c$$

$$\Rightarrow \int \frac{1-\sin v}{1-\sin^2 v}\, dv = x + c$$

$$\Rightarrow \int \frac{1-\sin v}{\cos^2 v}\, dv = x + c$$

$$\Rightarrow \int (1-\sin v)\cos^2 v\ dv = x + c$$

$$\Rightarrow \int (1-\sin v)\sec^2 v\ dv = x + c$$

$$\Rightarrow \int \sec^2 v dv - \int \sin v \sec^2 v\ dv = x + c$$

$$\Rightarrow \int \sec^2 v dv - \int \tan v \cdot \sec v\ dv = x + c$$

$$\Rightarrow \tan v - \sec v = x + c$$

Using the value of v back, we get

$$\Rightarrow \tan(x+y) - \sec(x+y) = x + c$$

is the required solution.

EXTRA EXAMPLES :

(1) Obtain the order and degree of the following differential equation.

(a) $x + \frac{dy}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

(b) $y = x\frac{dy}{dx} + 3\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

(c) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + e^x = 0$

(d) $\sqrt[3]{\frac{d^2y}{dx^2}} = \sqrt{\frac{dy}{dx}}$

(e) $\frac{d^2y}{dx^2} + \sin(\frac{dy}{dx}) + y = 0$

(f) $(\frac{d^2y}{dx^2})^2 + (\frac{dy}{dx})^3 + logy = 0$

(2) Verify that $y = x^2 + cx(c$ is arbitrary constant) is the general solution of the differential equation $xy' = x^2 + y$.

(3) Verify that $y = (x+c)e^{-1}$ is the general solution of the differential equation $\frac{dy}{dx} + y = e^{-x}$,where c is arbitrary constant.

(4) Find the differential equation of the following family of the curves , where a and b are arbitrary constants :

(a) $x^2 + y^2 = a^2$

(b) $x^2 - y^2 = a^2$

(c) $\frac{x}{a} + \frac{y}{b} = 1$

(d) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
(e) $(y - b)^2 = 4(x - a)$
(f) y $= ax^3$

(5) Solve the following differential equations . Also find the particular solution where the initial conditions are given .
(a) $(1 + x^2)dy = xydx$

(b) $y(1 + e^x)dy = (y + 1)e^x dx$

(c) $5\dfrac{dy}{dx} = e^x y^4$

(d) $x\cos^2 ydx = y\cos^2 xdy$

(e) $xdy + ydx = xydx, y(1) = 1$

(f) $xy\dfrac{dy}{dx} = y + 2, y(2) = 0$